The Mathematics Teacher

APRIL 1956

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Building a mathematics program—an adventure in co-operative planning[1]

FRANK B. ALLEN, *Lyons Township High School, La Grange, Illinois.*

This account of an "adventure in co-operative planning" underscores a fundamental principle of group action; namely, any administrator or committee that works on a problem involving groups of people must give first consideration to means whereby the group will accept the problem as its own.

IT OCCURS TO ME that there are several ways in which the title, "Building a Mathematics Program—An Adventure in Co-operative Planning," might be misunderstood.

It might get a negative response from some ultra-conservative people who believe that there is no need to modify the traditional curriculum in mathematics at the secondary level. This position is sometimes supported by the idea that mathematics never changes, that 2 and 2 always make 4 and that our only problem is to get the immutable facts of mathematics across to each new generation. I think teachers realize that the situation is not that simple. Profound changes in the content and organization of secondary mathematics appear to be inevitable. Well-informed teachers are becoming increasingly aware of this need for change.

Teachers evaluate these demands for change in terms that are both conservative and discriminating. However, we must realize that there was never a time when mathematics was more dynamic than it is today. There was never greater need to examine the organization and content of secondary mathematics in the light of changing conditions. In fact, there are times when the distance from where we are to where we think we should be seems so great that we despair of ever getting there.

I am reminded of a story about a traveler who became lost in a remote section of Tennessee while trying to find his way to Chattanooga. He finally asked a farmer for directions. After several determined but unsuccessful attempts to describe the route, the farmer gave up with the curious statement, "If I were going to Chattanooga, I wouldn't start from here." My point is, we *must* start from "here" in the sense that any recommendation for improvement in our mathematics program must be based upon a careful evaluation of what we have now. The major theme of this article is that co-operative study by classroom teachers should play an important role in such evaluations and in the development of recommendations based thereon.

Some of you might agree to all this and still want to question the word "adventure." After all, co-operative planning by classroom teachers can mean many different things.

For some it evokes the picture of perspiring teachers, working night after night in forlorn and desperate efforts to develop

[1] An address given at the Luncheon Meeting of The National Council of Teachers of Mathematics at Bloomington, Indiana, August 23, 1955.

instructional materials, the only virtue of which is a certain dubious originality. This could hardly be called adventure. For others, it means an emerging curriculum where change is mandatory in accordance with every new plan for "reconstruction," "adaptation," "integration," and the like. In some of these plans teachers work endlessly as on a treadmill to complete plans that are endlessly revised. Surely this is frustration, not adventure. For still others, curriculum work has turned out in practice to be primarily a matter of in-service training. The purpose is not so much to improve the teaching situation as to improve the teacher. Needless to say, there are a good many teachers who can control their enthusiasm for this kind of activity. I will be the first to agree that none of these activities can be described as "adventures in co-operative planning."

I think we all realize that there are many ways in which a so-called planning program can become an unjustifiable drain on the teacher's time and energy. The teachers may be exploited, while little if any benefit accrues to the school.

On the other hand, it does not have to be this way. There are conditions under which curriculum building can be a great adventure for the professional man. Such is the case, for example, when a teacher participates in developing a policy that is later put into effect, and he sees it operate to the benefit of the school. This experience conveys a sense of accomplishment, which makes all the effort seem worthwhile.

As chairman of the Steering Committee of the Lyons Township High School Curriculum Survey, I have been sharing such experiences with all of the members of the faculty of that school. I should hasten to inform you that I am not an authority in the field of curriculum development. I am primarily a teacher of mathematics who has just returned from an excursion into this broad field. I continued to teach three or four classes each year throughout the five year period (1950 to 1955) required for the completion of a comprehensive survey, which involved a total of thirty-three reports.

In order to tell you about our adventure in co-operative planning in mathematics, I must first tell you something about our school and community, and I must tell you a little about the Curriculum Survey of which the mathematics study is a part.

Ours is a township high school in a suburban area about 15 miles west of Chicago. Our enrollment has increased from 1,800 to 2,400 in the last five years, and our faculty has increased correspondingly from 96 to 120. We expect even greater increases in the next five years. About 75 per cent of our graduates go to college and about one-quarter of these go to our own junior college, which is housed in the same building with the high school. All of our staff members are experienced, and most of them have Masters degrees in their subject matter field. Since there is little turnover in personnel, it is relatively easy to achieve continuity of policy through the use of faculty committees.

Perhaps this is enough to give you some idea of the setting for our survey. You may be wondering how it happened that I, a classroom teacher, should be operating as chairman of a comprehensive survey of this kind. The reason is one of the things that makes our survey unusual. I think you should understand this reason if you are to understand why our entire faculty was able to sustain interest in the survey for so long a period. It all came about as the result of an idea advanced by our Superintendent. In 1950 he suggested that the resources of our staff should be mobilized for the purpose of improving our curriculum. Acting on this suggestion, the Board of Education initiated this study with the express condition that it should be a self-survey, executed and controlled by classroom teachers.

As it worked out, the control was exercised by a Steering Committee whose 22 members represented every department and service in the school. This committee

was instructed to conduct an across-the-board survey for the two-fold purpose of evaluating our school and developing recommendations for improvement based on that evaluation. After five years, the committee concluded this task. All reports have been written, and many recommendations have already been put into effect.

Before I begin my discussion of the "Mathematics Report," I must tell you a little about our procedures, so that you will understand the three steps we used in the development of each report. In the first stage, we were concerned with evaluation; therefore, we wanted to get the viewpoints of teachers from other fields. In order to accomplish this, the original committee was composed of "insiders" and "outsiders" in the ratio of about 3 to 1. The insiders were members of the department being surveyed, and the outsiders were chosen from other departments. In each case one of the outsiders served as chairman, and the chairman of the department being surveyed was assigned elsewhere. The Steering Committee believed that this arrangement was necessary in order to enhance the integrity and validity of our self-evaluation. The committees were told that this was to be an unbiased investigation and not merely a recital of strong points such as one would expect to stress in a publicity release.

In the second stage, we were primarily concerned with organizing our report and with the consideration of tentative recommendations in terms of the entire school situation. This step was performed by the Steering Committee.

In the third stage, we were concerned with verifying our recommendations and with finding ways of putting them into effect. This step involved review by the entire department with the department chairman presiding. With this arrangement, all findings and recommendations were thoroughly reviewed before the final draft was written. The whole process might be called "building of consensus through multiple review." This plan takes time, but it insures that every person in the group directly concerned with a decision will have some part in making it. Moreover, it creates an atmosphere of understanding, which makes the ultimate adoption of recommendations a virtual certainty.

Now that we have a fair understanding of the philosophy and procedures that characterized our survey, let us focus our attention on the "Mathematics Report" to see how it developed and what it accomplished for our school. To do this, we must go back to the first phase of our study, where we were primarily concerned with evaluating and diagnosing our mathematics program. I must, because of space limitations, confine my discussion to a few of the principal findings made by the original committee. The ones I have chosen can be considered under four major headings:

1. *College preparatory program*—On the whole, we thought we were doing a good job for our college-bound pupils. A four-year sequence was provided, consisting of four semesters of algebra, three semesters of geometry, and one semester of trigonometry. In 1952 we found that 90 per cent of our graduating class had taken two or more years of mathematics, while 50 per cent had taken three or more years and 20 per cent had taken four years. There was considerable evidence to indicate that our students were achieving reasonable success in their study of mathematics. Moreover, a comparison of high school and college grades by the National Registration Office demonstrated that our graduates were achieving about the same degree of success in college as they were in their work with us.

All this was very encouraging, but the committee was not the least inclined to complacency. Among other things, the initial report called attention to:

(a) A need for more effective co-ordination between instruction of mathematics and instruction of science,

particularly during the junior and senior years.

(b) A need for continued effort to maintain effective articulation with both grade school and college programs.

(c) The problem posed by those pupils who perform poorly in the college preparatory courses. Is their low achievement due to lack of effort, or should a different sequence of courses be provided for them?

(d) The lack of an honors program to provide acceleration for the more able college-bound pupils.

Later in this report, I shall describe some of the plans that were developed for dealing with these problems. Right now, I will continue with the other principal findings made by the original committee.

2. *Individual differences in ability and need*—It was evident that the department had given this problem considerable attention prior to the survey. A three-level homogeneous sectioning system extended through the first three years of the college preparatory sequence.[1] The committee found that this arrangement was working effectively. However, they believed that there were some pupils whose needs were not being met. Course offerings outside the college preparatory program were limited to one year of general mathematics for Freshmen and a one-semester course in consumer mathematics for upperclassmen. The committee was not satisfied with our provisions for individual differences beyond the ninth year, as is indicated by this brief excerpt from their report: "The difficulty lies in the fact that no provision is made for the slower students other than allowing them to enroll in the minimum sections of the sequence described above. No course sequence is provided for them. It is agreed that they accomplish little in the "M" sections of the college preparatory courses. In fact the very existence of "M" sections in a course such as Higher Algebra is regarded by some as an indication that adequate course provisions have not been made." Here again, the committee pointed out a major problem.

With regard to remedial instruction, the committee said: "Instruction of remedial character is provided to some degree in all classes. Nevertheless, we found a need for additional remedial instruction at the freshmen level for those who cannot succeed in general mathematics."

Provisions for developing the abilities of superior pupils were found to consist mainly of placing them in "S" sections, where valuable supplementary topics can be woven into the course. However, the committee pointed out that we had no special-projects programs for such students on a department-wide basis, and that there was no extra class activity related to mathematics.

3. *Evaluation of student achievement*—The departmental testing program was found to be excellent. Departmental examinations designed to test mastery of the fundamental ideas were given to all ability groups, with the result that grades assigned to individuals were consistent and defensible, regardless of the ability group involved. However, the committee expressed dissatisfaction with the fact that there were no provisions in operation at that time for student self-appraisal. It was believed that such procedures might do much to minimize the evils of grade chasing. When I mention grade chasing, I do not wish to convey the idea that all of our pupils are "eager beavers" in quest of high grades. We all know how difficult it is to steer a course between the Scylla of complete apathy on the one hand and the Charybdis of grade chasing on the other.

Many educators have suggested that the pupils' efforts to get a grade as "a badge of success" may seriously impair his ability to learn. A pupil so motivated

[1] Students were placed in "minimum," "regular," or "superior" sections respectively, according as their aptitude and achievements in mathematics were judged to be below average, average, or above average for our school. It should be noted that these sections bore no labels as far as pupils were concerned, but on administrative records they were designated by the letters "M," "R," and "S."

may tend to rely mainly on memory rather than on understanding. In our view the solution to this question does not lie in abolishing grades, but rather in assigning them so that they are valid indicators of the degree of mastery the pupil has achieved. The kind of mastery with which we are concerned cannot be achieved unless the pupil is able to recognize points of strength and weakness in his own work. It is for this reason that the committee suggested the development of a program for guided self-appraisal as an integral part of the teaching process.

4. *Time allotments*—Two major problems emerge as a result of the initial evaluation.

(a) The wisdom of devoting a full semester to solid geometry was sharply questioned. It was suggested that the more worth-while topics in solid geometry could be absorbed in other courses.

(b) The one semester allotment for higher algebra was held to be inadequate for the average pupil taking the course. So much time was spent in review that little was left for the study of advanced topics.

These are some of the salient points in the self-evaluation we made with the help of a few outsiders. It is my considered opinion that we were much more critical of ourselves than any outside agency would have been. From my vantage point as chairman of the Steering Committee I could see this same kind of frank, penetrating appraisal developing in every other department. In all cases our teachers were earnestly determined to find the weak spots and correct them. Judging by our experience at La Grange, we need not fear that teachers will be cynical or complacent in regard to self-evaluation. They agree with Carlyle that "the greatest of faults is to be conscious of none."

The next step was a review by the Steering Committee. In their written review, which was rather elaborate, the Steering Committee called upon the department to recommend policies and procedures, which would meet the needs set forth in the original report. They said in effect:

> Make more satisfactory arrangements for your slower pupils beyond the ninth year. Eliminate "M" sections in the college preparatory program.
>
> Improve remedial instruction for those unable to cope with general mathematics.
>
> Provide some kind of supplementary activity to challenge your superior pupils.
>
> Consider what can be done to encourage self-appraisal by pupils.
>
> Make more adequate time allotments for the study of algebra at the Junior level. Decide what you are going to do about solid geometry, etc., etc.

Before I describe the outcome of all this let me take a moment to point out the importance of the Steering Committee review in our scheme of things. It sharpened the issues, it improved the organization of the report, and most important of all, it provided a mandate for action. As members of the mathematics department read the review, they were conscious of a sense of mission, I might even say a sense of urgency, in dealing with the problems listed above. This interplay between the Steering Committee and the department was one of the essential conditions for the success of our curriculum study.

At this stage the entire department worked on policies and recommendations under the leadership of the department chairman. As we have seen, the assignment is very definite, and it is based on a thorough analysis of the existing situation.

In the remaining time I will describe some final outcomes as they were developed by the department, approved by the Steering Committee, and subsequently put into effect.

In order to make more satisfactory provisions for those pupils whose needs are

not met by the college preparatory sequence, we have extended our second track to three full years. The first year is fundamentally arithmetic in a general mathematics setting. It also contains an introduction to algebra through work with formulas. The second year is mainly algebra with emphasis on maintenance and extension of skills in arithmetic.

These two courses are regarded as equivalent to ninth-year algebra. As a matter of fact, we believe that the selected pupils who complete these two courses know a great deal more algebra than they would know if they had taken the regular course in Freshman algebra. Some of these pupils register for plane geometry at the Junior level. Some drop mathematics at this stage, while others continue with the third year of the second track program, which is being offered this year for the first time. We are beginning with two sections. The course is primarily plane geometry with an intuitive approach. There is a great deal of emphasis on computation and on the application of geometric principles to the solution of practical problems. We do not know yet whether this credit will be acceptable for college entrance. We believe, however, that these selected pupils will learn more geometry than they would have learned if they had taken our regular course in geometry, where a serious attempt is made to convey an understanding of deductive proof.

We do not regard this three-year second track as a model for other schools to follow. There are doubtless other schools whose second track programs are more extensive and better established than ours, but for us at this time our program represents a substantial advance and one which is directly attributable to the survey. Through the establishment of these courses, we have eliminated the anomaly of the "M" sections in the college preparatory sequences. This is heartening proof that committees are not always futile. Moreover, the entire department has a proprietary interest in this second track program, since everyone participated in the deliberations which led to its establishment.

In order to improve remedial instruction for those unable to deal with general mathematics, we now offer a course called Mathematics Fundamentals. You might well ask if the department can expect to accomplish anything worth-while in a course of this kind. All I know is that it has been a great experience for the pupils in this one class to work with a tolerant, skillful, patient teacher who understands them and is interested in them as individuals. This kind of work can go far toward removing their allergy to education, even if it cannot increase their capacity for it. On these terms we regard the course as a success. The initiation of this course is in line with the policy stated by the Steering Committee after studying the entire school situation, namely, "Departments which deal with educational values which are essential to all youth will make provisions for all youth within the course offerings of the department, with special attention to individual differences. No department is entitled to dump its problem cases on other departments."

In order to provide supplementary work for our superior pupils, we have experimented with mathematics seminars, where various members of the department meet with interested pupils after school to discuss advanced topics. Other provisions include the organization of a Mathematics Club and participation in mathematics contests sponsored by the MAA. Pupils are also encouraged to work on problems proposed in the *Mathematics Student Journal*. These measures add up to substantial progress, but we still feel that there is much to be done in this area.

In order to make more adequate provisions for individual differences in algebra at the Junior level, we have established a full year course called Intermediate Algebra for those who need more time to assimilate essential algebraic skills. We

have retained our one semester course called Higher Algebra for the benefit of superior students. The year course has proved to be very successful in meeting the need of the average pupil at this level. Looking back on all this, we can say that as a result of our survey provisions for individual differences have been substantially improved.

Solid geometry is still offered for the superior pupils during their second semester of the Junior year. We may eventually replace this course or at least offer another which can be taken instead. Frankly, at the present time there is little agreement among us on what kind of a course this should be.

Now we come to the problem of promoting self-evaluation by the pupil. We have a plan for accomplishing this, which was discussed at the time of the survey. Since then, it has been tried by several of our teachers. It is known as the "bid system." I would like to describe it for you. If any of you decide to try it, I would certainly be interested in your observations. It is essentially a very simple plan.

The pupils are provided with a list of problems and are told to study the list with the understanding that they will be required to solve certain problems selected from it under quiz conditions. In the meantime, the instructor agrees to help them with the analysis of the problems and to answer their questions about methods and procedures. He will not present complete solutions nor will he consume class time with lengthy computations. In a properly constructed list, as in a properly constructed examination, the number of problems a student can solve should determine his grade. We ask each pupil to state in writing how many problems he can solve and which ones. This statement is his "bid." The following day we ask him to verify his bid by solving two or three problems chosen from those he said he could solve. It is best to have the work done on the same paper that bears his signed bid. If he succeeds in solving these problems, he is credited with the number of problems shown in his bid, and this number determines his grade in accordance with a prearranged scale. If he fails on some of the problems, his credit is reduced, and his grade is adjusted accordingly. In no case is he credited for more problems than he claimed in his bid. In evaluating these solutions, credit is given for analysis and method as well as for numerical accuracy.

We have found that this plan offers a number of advantages to both teacher and pupil. For the teacher, it reduces the amount of time necessary for checking vast quantities of homework of doubtful origin. The pupil's time is also conserved because he is not expected to waste time writing out problems that he knows perfectly well how to solve. Instead, he is encouraged to concentrate his efforts on those problems which offer some challenge and thereby provide worth while learning experiences. In this sense the bid system automatically adjusts the assignment to the ability level of each pupil. One might think that a pupil could score well by merely memorizing solutions. In practice we find that memory will not carry a pupil very far toward the mastery of a comprehensive list of miscellaneous problems. I should also emphasize the fact that we are not encouraging the pupils to gamble. On the other hand, we are not inclined to moralize about it. The pupil knows that he will be asked to solve the most difficult problems we can find among those he contracted to solve, and he must be guided accordingly. He soon learns that honesty is the best policy.

We have found that this system serves to encourage some rather intensive self-evaluation by the pupil. He must at least locate his difficulties in order to capitalize on the question sessions. Moreover, he learns the importance of correctly estimating his own abilities. If he is over cautious, he has no chance for a high mark. If he over-bids, he is heavily penalized. Perhaps you will agree that, in this

regard, the bid system closely parallels certain life situations. I recommend it to your attention as a satisfactory method for encouraging self-analysis and as a valuable supplement to other methods of evaluation.

We have accomplished many other things in our survey, which I would tell you about if space permitted. I think we all agree that a feeling of accomplishment is essential to effective staff work in any field. It is invariably accompanied by a feeling that there is much that remains to be done.

For example, one day we were considering the fact that some students are lazy, disorganized, and unmotivated when one member of our group asked a very thoughtful question: suppose that by some miracle all of our pupils should suddenly become afflicted with an insatiable appetite for learning—with a consuming curiosity about the whys and wherefores of mathematics—what then? After an awkward silence, this led quite naturally to the question of what we should do for our superior pupils. Could they not go much farther if given a chance? There is a body of informed opinion which holds that we could modify our course offerings for our brighter students so as to provide them with a far better introduction to the study of modern mathematics and modern science. Saunders MacLane, writing in the May, 1954 issue of the *Bulletin of Secondary School Principals*,[2] says flatly that "the lively modern development of mathematics has had no impact on the presentation of secondary mathematics." We must try to find out what impact it should have on high school mathematics.

According to the Joint Commission on the Place of Mathematics in Secondary Education, "to mathematize a subject does not mean merely to introduce into it equations and formulas, but rather to mold and fuse it into a coherent whole, with its postulates and assumptions clearly recognized, its definitions faultlessly drawn, and its conclusions scrupulously exact."[3] Some of our leaders in mathematics and mathematics education believe that it is time for us to try to make some progress toward mathematizing algebra at the secondary level. It is suggested that with this approach algebra ceases to be just a bag of tricks, or as Henry James says, "a low form of cunning," and becomes, instead, a proper introduction to the nature of mathematical thought.

I suggest that all these things need looking into, and furthermore that much of the work must be done by the cooperative efforts of classroom teachers. Only thus can they attain their full stature as professional men and women. To sustain this view, I appeal to the definition stated about twenty years ago by the international scholar Harold Laski[4]:

> A profession is a vocation in which,
> 1. Public service is more vital than private profit.
> 2. There is the profession of an essential idealism.
> 3. Its members enter it upon the basis of approved standards of competence.
> 4. Its members freely perform research on their problems to the common advantage.

In our case I interpret this to mean that we are to study our problems to the advantage of our pupils and ourselves. The performance of such studies is our privilege and our professional perogative as well as our obligation and responsibility.

When we are expected to do such work, we are not being imposed upon. On the contrary, we are being given professional status, and we are being recognized as people whose opinions are valuable and on whose judgment policies can be founded. In this regard I should mention that our

[2] Saunders MacLane, "The Impact of Modern Mathematics," *The Bulletin of the National Association of Secondary School Principals* (May 1954), 38: 66–70.

[3] "The Place of Mathematics in Secondary Education," *Fifteenth Yearbook of The National Council of Teachers of Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University), 1940, p. 8.

[4] Harold Laski, "Decline of the Professions," *Harpers Magazine* (November 1935), 171: 676–685.

Superintendent's sustained faith in the judgment of his staff was one of the principal reasons for our success. In effect he called the classroom teachers to full partnership in the making of educational policy.

Therefore, let us learn to work together on the solution of our problems. Let us try to master this intricate business of building consensus. This process can easily be misunderstood. It would be naive to suppose that all disagreement can be eliminated. It should not be eliminated, but the area of agreement should be maximized by discussion and study and through the reduction of semantic barriers to understanding.

Above all, let us not be too tired to meet this challenge. Let us not be so preoccupied with routine that we neglect this great opportunity. As we work together with a full realization of the importance of what we do, as we work in an atmosphere of mutual understanding of the kind that I have tried to describe, we will fully realize that building the mathematics program can indeed be an adventure in cooperative planning.

Summer institutes for high school and college teachers

During the coming summer the National Science Foundation will sponsor three institutes for teachers of mathematics. A limited number of stipends to cover expenses will be available for participants. Readers of THE MATHEMATICS TEACHER who are interested in one or more of the institutes should address their inquiries to the directors of the institutes.

Iowa State Teachers College. This institute for high school teachers will be held during the period June 18 to July 27 under the direction of Dr. Henry Van Engen, Department of Mathematics, Iowa State Teachers College, Cedar Falls, Iowa.

University of Michigan. This institute for college teachers will be held during the period June 25 to August 14 under the direction of Dr. T. S. Hildebrandt with the assistance of Dr. P. S. Jones, Department of Mathematics, University of Michigan, Ann Arbor, Michigan.

Williams College. This institute for college teachers and secondary teachers will be held during the period July 2 to August 11 under the direction of Professor Donald E. Richmond, Department of Mathematics, Williams College, Williamstown, Massachusetts. The principal speakers will be Dr. B. W. Jones of the University of Colorado and Dr. S. C. Kleene of the University of Wisconsin.

Psychology of learning in the junior high school

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Too often mathematics is "taught" without regard for those things which are known about the learning process and the characteristics of a particular age group. This article summarizes some of these important facts for the junior high school teacher.

THE STUDY of human behavior has brought forth an accumulation of facts regarding intelligence that enables us, under proper use, to propose challenging material for instruction. Behavior has many aspects—social, emotional, physical, and mental—and in genuine educational situations all of these aspects must be considered. However, we consider here only those psychological aspects related to intelligence.

It is recognized that the total past experience of a child, the environment in which he has lived and the type of teaching to which he has been subjected before he enters the junior high school, all affect the learning that can and will take place. Nevertheless, from the ages of 11 years to 15 years human beings make continuous and considerable growth in the handling of relationships dealing with space, time, and number, in making comparison of quantities, in memorization of essential data, in acquisition of new mathematics vocabulary, in using imagination and creative ideas, in abstracting qualities to make generalizations, and in ability to concentrate on the solution of problems. To support these assertions let us look at the findings of research concerning growth of mentality in human beings.

First, learning ability increases at an increasing rate from the ages 11 to 15 years. The *memory span* increases regularly with age. Studies carried on by W. H. Pyle (1)[1] show that the ability to read prose for ideas and to immediately recall these ideas is far greater at age 14 than at age 10 years. Likewise, the ability to look at a configuration (picture) and to recall the items in the picture grows in increasing amounts each year from age 10 to 14. In a study of the time required to learn new words and to memorize numbers the results confirm the fact that each year of age shows a smaller required interval of time. Children learn faster as they grow older from ages 10 to 14. There is no letup during the junior high school age of both boys and girls in their ability to have an experience in mental activity and recall the experience. In fact this ability grows each year with an increase in storage capacity of the learning and with less time requirement for making it. The experiments show this to be true whether the material to be learned is meaningful or nonsense, i.e., with relation to previous knowledge or by pure rote. However, Ebbinghaus (2) and others as early as 1880 have shown that meaningful learning is acquired in the least time and has greater permanency for recall.

In learning *perceptual-motor skills* Pyle (1), through a series of studies, found a

[1] The numbers in parentheses refer to the bibliography at the end of the article.

similar result. Whether it be card-sorting, alphabetizing, mirror drawing, space imagery, or drawing and measuring with instruments, the time required to learn the skill decreases with age. There is no let-up from early age (8 years) until late adolescence (21 years). The age group 10 to 14 years grows in its ability to use tools. The junior high school age should show great growth in the use of rulers, protractors, compasses, and other mathematical instruments.

A higher mental process is that of *grasping relations*. Heidbreder (3) has done outstanding research on the ability of humans to solve a problem by having them discover a principle of recognition through errorless choice. A series of pictures are presented that can be identified as belonging to a particular named class through the recognition of a common property. These pictures are repeated in different forms and sequences until the identification is learned. Children from ages 6 to 10 years, 11 to 15 years, and university students were tested. The rate of learning increased with increasing age. Children grow yearly in their capacity to grasp relations and solve problems.

In connection with this type of learning, studies have been made on the *effect of overlearning*. Overlearning is more thorough learning than is required to perform well on an immediate test. Thus review and drill constitute procedures for overlearning. All experiments prove that overlearning aids greatly in retention and future recall. Review overlearning is most effective when spaced so as to come when some skills are slipping away through lack of use. The best type of review (for retention) appears to be that which incorporates newly learned materials with the old reviewed material. This aids integration and organization of the learning.

As children grow older they change their approach to learning. The *types of learning* assume a higher and more mature form. Children in the elementary school use a trial and error approach to learning that takes much time. In the junior high school the children give greater stress to making analyses of problems and bring about greater reorganization of past experience. (The arithmetic becomes an integrated body of knowledge, not whole numbers and fractions and decimals.) In the senior high school the learning takes on the adult form of circumspection and delayed attacks until the whole problem is studied. This is shown by submitting puzzles to the individuals and watching their manner of solution. The junior high school is the transition stage from immature hit-or-miss trial to the mature analysis and solution-seeking type of learning.

In this transition, transfer of past training is playing a continuously greater role. Under proper instruction, the problem-solving techniques will be headed toward the mature gestalt-insightful learning procedures. The children from ages 11 to 15 years are making this change. They become more cautious, more patient, more systematic, and more organizational in their learning. The teacher is the important guide post. This change in approach to learning has great implications for concept learning and problem solving. It also calls for a rather solid and well-organized knowledge of high school and early collegiate mathematics on the part of the teacher, so that correct mathematical concepts are developed. The transfer and the ability to transfer past knowledge to new situations is dependent on the manner of teaching. "Practice at learning keeps active one's methods of study and yields an increasing body of potentially transferable knowledge and may indirectly increase motivation to learn." Failure to practice learning may be a deterrent to continued learning in the senior high school.

The greatest deterrent to retention and learning is *interference*.[2] Interference is a

[2] When a student's attention is distracted so that he is not attending the instruction, we do not think of this as interference. This is removal from the learning situation.

mi cing up of ideas. It occurs when a first or primary idea is not completely understood before proceeding to new ideas or when several unlike ideas are introduced simultaneously. Thus to introduce the solution of equations of the type $2x=6$ (solved by division) and $\frac{1}{2}x=6$ (solved by multiplication) at the same time may cause interference. At best, if both types of equations are introduced, it would be better to use only one idea, namely, divide by the coefficient of x. As children grow older, a reorganization of responses causes difficulty in learning. Thus, the type of learning that requires the least reorganization is the easiest. In this connection many *new* geometric relations that are first-time experiences are easier to learn than the theory of percent and percentages, where a reorganization of earlier work in arithmetic to new language forms is necessary.

In summary, there is sufficient evidence to prove that there is no let-up from ages 10 to 15 years in growth in mental ability or intelligence as related to memory, perceptual motor skill, concept formation, making of generalizations, and problem solving. Some recent studies on academic achievement have indicated a plateau or leveling off period in acquiring subject matter is reached during the ages 11 to 14 years. Upon the psychological evidence presented it is the judgment of the writer that this must be ascribed to a nonchallenging curriculum or to sterile teaching and learning situations. The correlations between age and rate of scholastic learning are all over .75 and this means no plateau in academic achievement should exist.

Learning as a function of test intelligence

We know today that in the realm of academic study the best indicator of ability is *verbal facility*. The ability to use and understand words, to communicate, is far removed from rote learning. In fact, the data from studies by Thorndike (4), Conrad (5), and Jones (6) on *Observation and Recall as a Function of Age* suggest that *meaning* is the one important dimension determining the correlation between learning and intelligence. *Materials learned under instructions to reproduce meanings* correlate more highly with intelligence than these materials learned by rote. It was also evident that regardless of the type of instruction there is a higher correlation between learning and intelligence to the degree that materials permit attack by a logical method. This relation of meaning and intelligence was sustained in a very recent research by Dr. Max Sobel (7) on learning ninth-grade algebra.

To distinguish between *bright and dull learners* Wilson (8) in 1928 subjected students of high and low intelligence to various types of learning with the following results. *Memory* learning of a multiplication table and shorthand characters yielded the clearest difference. Brightest children have the best memory capacity. Next came reproduction of *meaning* in a paragraph and discovery of a general principle. Simple motor tasks yielded very little difference. Permanent learning of meaningful mathematics is highly correlated with intelligence. Hollingworth (9) in the study of very bright children (IQ 146 to 165) found that the hierarchical order of experiences distinguishing intellectual capacity was: first, abstracting meaning from sentences and paragraphs; second, the study of science; third, learning the meaning of addition of fractions and multiplication of integers. In any learning situation as the number of elements, the complexity, or the difficulty is increased, the difference between good and dull learners became more apparent. Most teachers have discovered this in their own experience. A synthesis of these studies on learning indicate that the five dimensions to learning beginning at the top are: (1) meaning; (2) amount of symbolic process; (3) complexity and difficulty of relations; (4) amount of discovery required; and (5) muscular response. The

first four dimensions are significant, but the last is barely related to test intelligence or intelligent action.

Other studies on learning show that a learner can accelerate his *rate of learning* by making a *search for meaning* by the imposition of rhythm or a pattern (by structuring), by making new groupings of items in a situation, by noting spatial relations, and by devising organizations of materials learned. By making a new situation more meaningful he thus assimilates it more readily into his already existent patterns of response. *The "logical method" is far superior to the "rote method" in learning new knowledge.*

Motivation

Perhaps the most modern and the most important psychological aspect of intelligent learning is *motivation based on needs.* Much has been done on this phase of learning both philosophically and experimentally, and the literature is vast. We shall report here only that which appears significant for learning in the junior high school. Symonds (10) has given an interpretation of Maslows' hierarchical order of needs of human beings as follows:

a. Objects and persons to satisfy inner physiological needs
b. Safety—escape from outer dangers
c. Acceptance, involving affection, affiliation, belonging, companionship
d. Approval, prestige, status
e. Self-esteem, self-respect, self-satisfaction
f. Mastery, success, achievement
g. Independence

The first three needs are basic early in life. It is the last four needs that rise in junior high school boys and girls. All these needs involve the self, that is, the individual as a person. Whatever satisfies these needs becomes the goal of the individual. Motivation is goal-directed behavior. To motivate a pupil, present him with goals to be attained, which when attained will satisfy the needs listed in "d" to "g" above. If we accept these needs as basic, then the self of the student is most important. Motivation for most human learning resides in the inter-personal relationship of the student with other students and with his superiors (teachers and parents). The most basic incentives to learning a teacher can give are "acceptance," "approval," and "respect." Travel from convention to convention and seek the reason for the great things mathematicians are doing, and you will find that mastery, success, independence, recognition, and approval are the greatest drives to learning and creating—not simply the love of mathematics itself, as great as this may be.

A pupil in junior high school will continue to learn or to be discouraged from learning to the degree that his expectations or goals have been reached. The influence of success is a tremendous incentive. An experiment has been performed in which students were given problems to be solved and then, after a given time, urged to solve the next one. Some of the problems were not possible of solution by the students. Success led to heightened interest and eagerness to do the next one. The reaction to immediate failure was hard to estimate, but when three or four insolvable problems occurred in a row, there was practically no incentive to continue. It thus appears that intermittent rewards (success) seem most favorable to motivating further learning. Developing self-confidence is an incentive to learning. Best of all, enthusiasm of the teacher for his subject seems to be a major factor in motivating his students to learn.

Competition has long been held one of the key incentives to further learning. Recent investigations throw much doubt on this hypothesis. Among the major findings are the following (11):

1. It violates the agreement that learning is possible only if the goal is attainable. When a whole class of heterogeneous students compete for the top prize,

there is motivation only for the very best in the class, for only the best can attain the prize or reward by winning; for the others there is only defeat. Thus competition, if used continuously, makes the strong stronger in learning, and the weak, who need more incentive than the strong, benefit not at all.

2. Competition for high honors or even just to pass a high standard is not a genuine learning situation, in that the purpose is to excel and not necessarily to understand one's own processes. Only the excellent get the approval and self-esteem so necessary for learning. It damages the self-respect of the losers and interferes greatly with motivation for further learning.
3. Good competition is with one's own peers or with one's own prior achievement. This is competition that permits success as well as failures. It has been shown that continued success without any failure causes a lack of motivation as well as continued failure. Self-esteem comes with real achievement, achievement that has passed by difficult hurdles and not merely followed a rosy path with no troubles.

Along with competition, school grades, or marks, have long been one of the teacher's means of motivation. Now a high grade, A or 100, is frequently motivation to learning, not because of rewards, but because of the more basic satisfaction of approval, status, and success. We must remember that basic motivations reside in human relations, and that the striving for external rewards such as high marks or honors is subordinate to and hidden in the striving for direct recognition and approval. The boy or girl interprets a high grade to mean "you approve," and by the same token he interprets a low grade to mean "you disapprove" of him, you have cast him out, he has lost face. You may try very hard to tell these boys and girls that they get only what they earn, and the mark is what they have earned, yet the boys and girls will not listen to you. Yes, good grades motivate, but low grades discourage and do not motivate, and in neither case do the grades show how learning should take place. To be admired, approved, and accepted are basic to motivating learning.

Comparing the psychology of learning with the present curriculum in junior high school mathematics must raise some serious problems in our minds. Certainly we should have more challenging intellectual material because our pupils are not achieving to capacity, especially the more able students. Certainly we can well consider our methods of teaching, especially in light of the necessity and possibility of realizing better reading, understanding, and problem-solving ability. There would also be quite some doubt as to the procedures we are using to motivate and keep alive a burning desire to study mathematics, a very much needed goal of our future citizenry. The most immediate problem appears to be that of creating a body of mathematics, knowledge of sufficient complexity and abstractness to be commensurate with the growing intellectual capability of the students.

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College Entrance Examination Board announces a Commission on Mathematics

A Commission on Mathematics that will investigate the need for revision of the secondary school mathematics curriculum has been appointed by the College Entrance Examination Board. The Commission has been established by the College Board in recognition of a growing divergence between developments in modern mathematics and the type of mathematics taught in secondary schools.

For over 50 years high school mathematics for college-preparatory students has remained essentially unchanged—the traditional presentation of algebra, geometry, and trigonometry. Much readjustment of teaching methods and topical emphasis has of course occurred, partially under the impetus of recommendations of such able groups as the 1923 Committee on Mathematical Requirements.

Meanwhile, the very nature of mathematics has changed radically to a form now vaguely referred to as modern mathematics. After stressing modern mathematical concepts only in graduate courses for many years, many college mathematics departments are readjusting Freshman courses to include these ideas. No equivalent trend has yet developed widely on the high school level.

Curricular reform has often failed to result from the work of previous conferences and commissions in mathematical education because their recommendations have been only partially implemented. In view of this, the College Board is prepared to take further action. If there is general agreement as to the need and form of curricular revision, the College Board will, if necessary, foster the development of teaching materials and workshops to prepare teachers to utilize these materials.

The Commission met for the first time at Ann Arbor, Michigan in August 1955 and reached the following tentative conclusions:

1. High school mathematics is in need of revision, which would include the addition of new materials, and the elimination of some and the reorganization of other topics now being taught;

2. In considering a topic for deletion or inclusion, the Commission will bear in mind the relevance of the topic to modern mathematics, social and natural science, and engineering;

3. Liaison with and the co-operation of all groups interested in school and mathematics curricula is essential; and

4. The development of a single definitive syllabus is undesirable, if not impossible. The Commission therefore proposes to formulate a broad list of topics and to suggest where and how they might be introduced.

Members of the Commission include:

Professor Albert W. Tucker, Princeton University (Chairman)
Professor C. B. Allendoerfer, University of Washington
Mr. Edwin C. Douglas, The Taft School, Watertown, Connecticut
Professor Howard F. Fehr, Teachers College, Columbia University
Miss Martha Hildebrandt, Proviso Township High School, Maywood, Illinois
Dean Albert E. Meder, Jr., Rutgers University
Professor Frederick Mosteller, Harvard University
Professor Eugene P. Northrop, University of Chicago
Dr. Ernest R. Ranucci, Weequahic High School, Newark, New Jersey
Mr. Robert Rourke, The Kent School, Kent, Connecticut
Professor George B. Thomas, Jr., Massachusetts Institute of Technology
Professor Henry Van Engen, Iowa State Teachers College
Professor Samuel S. Wilks, Princeton University

The Commission welcomes the assistance and suggestions of organizations and individuals. Those interested in corresponding with the Commission should write to Mr. Robert Kalin, Executive Assistant, Commission on Mathematics, P.O. Box 592, Princeton, New Jersey.

An English schoolmaster looks at American mathematics teaching[1]

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An account of the insights into American education gained by one of our British "cousins," while teaching in the Denver, Colorado schools.

DURING THE SCHOOL YEAR 1950–51 I had the privilege of being one of the British teachers visiting the United States of America under the official exchange scheme. In addition I had the great good fortune to be assigned to Denver, Colorado, which must have as beautiful a situation as any city of its size in the world. Denver has a population of about half a million, and its school system is reputed to be one of the best "middle of the road" systems in the United States. I believe, therefore, that I had the pleasure of observing and sharing in American education at its best.

The first point I must make—and I cannot make it too forcefully—is that American education, like everything else in America, is diversified in the extreme. I cannot say with any certainty which of the things I saw are really typical of American education and which are peculiar to Denver or to the State of Colorado. I visited a number of schools—about fifty—in Denver and in different parts of the State of Colorado and talked with teachers from many parts of the United States, but it was by no means easy for me to separate variations of detail from variations in the pattern of education. Here, I shall simply give my impressions of what I saw and learnt in America and make appropriate comparisons with our own schools to point similarities and differences.

For any reasonable appreciation of American methods, some knowledge of the American educational system, which is very different from our own, is essential. There is a Federal Office of Education in Washington, D.C., but it is one of the minor government agencies, and its functions are chiefly statistical and advisory. Education is in the hands of the individual states; in educational matters the states are supreme. Since the states vary widely in population, industries, and wealth, there is a wide variation in their education laws—variation in leaving age, financial arrangements, teachers' qualifications, and so on. For example, the minimum leaving age varies from fourteen to eighteen; in Colorado the law is "when the age of fifteen is reached, or when the eighth grade has been completed." Most states maintain Offices of Education, but these have, it seemed to me, less influence on education in their states than has the Ministry of Education in Britain.

The units of educational administration are the school boards (comparable to the school boards which we had in the nineteenth century) that cater for limited areas, frequently with small scattered rural populations. By numbers, the commonest American schools are still all-age,

[1] Reprinted from *The Mathematical Gazette*, XXXIX (May 1955), 89–97, with the permission of the Editor and the author.

one- or two-teacher schools. On the other hand, the city of Denver has only one School Board, and the schools number their pupils in hundreds. The members of the school boards are directly elected for various terms, and their chief executive is termed a superintendent. The superintendent and his attendant administrators interfere a good deal more in the day-to-day life of the schools than do Chief Education Officers and their assistants in England. For example, it is quite common for courses of study and textbooks to be prescribed by the administration.

The schools are generally comprehensive and coeducational. By that I mean that all pupils of a given age in a given area normally attend the same school. This means that where numbers allow—that is, in the towns—schools are large. A common way of dividing the schools is: grade schools for pupils from six to twelve; junior high schools for pupils from thirteen to fifteen; senior high schools for pupils from sixteen to eighteen. But other modes of division are also used. Grade schools may have up to three or four hundred pupils; junior high schools up to one thousand pupils; and I heard of senior high schools with as many as ten thousand pupils. I myself taught in a senior high school with 2,500 pupils. In the grade schools the pupils have the same teacher for all or almost all subjects, as in our primary schools; in the junior high schools there is some subject teaching, as in our secondary modern schools; in the senior high schools all teaching is subject teaching, as in our grammar schools. Large schools are needed to provide an adequate range of courses to suit all intellectual levels and to attract the interests of the pupils. This is considered of great importance, and schools are partly judged in the public eye by their ability to attract pupils after the age of compulsory attendance.

American educational objectives are stated more explicitly, more precisely, and more frequently than are our aims. One popular statement of the goals of American education is headed, "Imperative Needs of Youth," and these are listed as:

1. All youth need to develop saleable skills and those understandings and attitudes that make the worker an intelligent and productive participant in economic life. To this end, most youth need supervised work experience as well as education in the skills and knowledge of their occupations.

2. All youth need to develop and maintain good health and physical fitness.

3. All youth need to understand the rights and duties of the citizen of a democratic society and to be diligent and competent in the performance of their obligations as members of the community and citizens of the state and nation.

4. All youth need to understand the significance of the family for the individual and society and the conditions conducive to successful family life.

5. All youth need to know how to purchase and use goods and service intelligently, understanding both the values received by the consumer and the economic consequences of their acts.

6. All youth need to understand the methods of science, the influence of science on human life, and the main scientific facts concerning the nature of the world and of man.

7. All youth need opportunities to develop their capacities to appreciate beauty in literature, art, music, and nature.

8. All youth need to be able to use their leisure time well and to budget it wisely, balancing activities that yield satisfactions to the individual with those that are socially useful.

9. All youth need to develop respect for other persons, to grow in their insight into ethical values and principles, and to be able to live and work cooperatively with others.

10. All youth need to grow in their ability to think rationally, to express their thoughts clearly, and to read and listen with understanding.

(Quoted from *Planning for American Youth*, published for the National Association of Secondary School Principals.)

It is clear that the goal of satisfying these needs provides a clear and intelligible philosophy of education, though there is obviously much room for variation in emphasis. Further, as in all educational projects, what is planned is not necessarily what is executed.

Clearly, mathematics as far as and including elementary arithmetic, knowledge of simple formulae, use of simple graphs, and a little stage A geometry is needed to meet several of these needs. Consequently all pupils follow courses which cover these fundamentals. Beyond these basic skills there is an elaborate system of options, satisfying the needs of pupils with definite professional or vocational aims, and one of the most interesting and suggestive things about American education is the way in which these options are organized and the way in which pupils are guided to exercise a wise choice among the many courses offered. In no case did I find the choices were made in a haphazard way, which many English writers on American education suggest. At a recent education conference in this country I heard it said that, "If a boy doesn't like algebra, he drops it and takes up something else, such as learning to drive." In Denver, at least, that is not so, for most senior high school pupils have driving lessons in school time as part of their general education program. Remember, in America almost everyone has a car, and correct handling of a car is a social responsibility.

The aim of each senior high school pupil is to gain a graduation diploma—what we should term an internal leaving certificate—and the requirements to be satisfied for such a diploma are carefully prescribed. For example, in Denver, during three years in senior high school, 15 units must be earned (a unit is a year's work at the rate of one period daily) which must include 2 units of English, $1\frac{1}{2}$ units of social science, 1 unit of mathematics, 1 unit of physical education, $\frac{1}{2}$ unit of guidance, $\frac{1}{2}$ unit of health, making a total of $6\frac{1}{2}$ units and leaving $8\frac{1}{2}$ units at the pupil's disposal. The pupil will normally have two major fields of interest, in each of which he will do 3 units of work. For a unit to count, the work must be done to the satisfaction of the teacher, and it is quite common for a teacher to refuse credit for a term's work which he considers unsatisfactory. Assessment is based on classwork, homework, performance in attainment tests, and general co-operation of the pupil.

Clearly the whole system depends on guiding pupils into appropriate courses. During his three years in senior high school, each pupil has a member of staff for counselor, with whom he has conferences at regular intervals. The counselor is the teacher who conducts his guidance class and frequently also conducts his social science classes. All teachers are encouraged to share in this work, and training in guidance is part of a teacher's professional preparation. Course selection is based on a consideration of the pupil's abilities and interests, his ambitions, his parents' wishes, graduation requirements, etc.; the decisions are mutual decisions, not orders from above. Changes of course are not undertaken lightly, but do occur. The most obvious effect of this method of planning is that the majority of pupils have a coherent conception of the purpose of their school studies and consequently require less urging to do the necessary modicum of distasteful toil. In fact, one of my strongest impressions on resuming teaching in Britain was how aimless many of our fifth form pupils are compared with their American counterparts.

After this survey of the American educational system as I found it, perhaps I

may turn to more specifically mathematical topics. Again the starting point must be course arrangements in a senior high school. Courses offered are of three types.

First, there are classes in remedial mathematics. These classes are organized under a variety of names, but all have as their objective the removal of defects in basic mathematical competence or understanding. Defects of this sort, I believe, are not unknown in English grammar schools! Recently I had a bitter argument with a fifth form pupil on the question whether the solution of the equation $x/3 = 6$ involved multiplication or division by 3! There appear to be two possible ways of dealing with these basic defects: one is to assume that previous teachers were negligent or incompetent and did not give the pupil sufficient practice; the other is to conclude that the pupil was presented with certain material when he had not sufficient maturity for that material, as presented, to have meaning for him. The first is, of course, the classical attitude and leads to drill and yet more drill; the second is the attitude of modern educational theorists. In America I was very much impressed by the excellent results being obtained in these remedial classes, by the equipment available to diagnose deficiencies and the care taken to identify the exact skills lacking, and by the trouble taken to remedy defects identified. But in spite of the enormous amount of research published on psychologically effective approaches and means of making mathematics meaningful, I always felt that the classical method was the one in use, and that the results of recent research were given little serious consideration.

The second type of class which is common is the general mathematics class. Ostensibly, the philosophy of these classes is the philosophy behind the recommendations of the Jeffrey Report; that is, that mathematics should be taught as a whole, integrating the different branches into a comprehensive subject, and that the subject should be made relevant to modern life and intelligible to the pupils. There is the same attempt to prune away dead wood and strip off formal manipulation and mechanical routines. In practice these classes often turn out to be arithmetic with a very slight admixture of algebra—substitution in easy formulae and a few simple graphs, with the intention of introducing the idea of functionality—and a little geometry—some mensuration. The claim is that these classes are for all pupils, but in practice it is usually the weaker pupils who find their way into them. Again, an excellent job is being done within the limitations accepted, and the teachers strive manfully to overcome the restrictions of the situation in which they find themselves. Nevertheless, these classes in practice seem to me to be rather different from those envisaged by their proponents.

The third type of class is the traditional formal mathematics class, which we know so well. But the American practice is to teach one branch of mathematics at a time. First, a year or a year and a half of a two-year algebra course; then a year of plane geometry, followed by half a year of solid geometry; the remainder of the algebra course; then half a year of trigonometry; finally, half a year devoted to analysis. In certain cases two of the later courses are taken concurrently, and there are proficiency bars at various stages. The classes move on solid traditional lines and concentrate on giving a thorough training in the basic techniques of the various branches of mathematics. In the algebra classes particularly, the subdivision of subject matter is not quite so complete as the titles of the classes seem to indicate, and some geometrical work is often included. Formal manipulation, however, is given more emphasis and is carried to higher levels of complexity than we are accustomed to today. Standards are reasonable, having regard to the age at which the studies are started and the time spent at them. If a subject is being

studied, it is studied for one period each day—timetables are designed on a daily basis instead of the weekly basis used in this country. This time allowance is exactly what we have up to the fifth forms, but is considerably less than we have in the sixth forms. The final level reached by the best pupils was rather higher than the subsidiary subject standard of the old Higher School Certificate. In a school of 2,500 about 10 pupils each year reached this level—say, 1 per cent of the age group. We might compare this with an equal number of pupils reaching advanced G.C.E. level in a three stream grammar school. The result seemed to me to be quite reasonable and compared well with what we find in the average English grammar school. Of course, many English grammar schools reach a much higher standard quantitatively, but their pupils are usually subject to some form of selection beyond the usual special place examination.

This raises the whole thorny question of comparative qualitative standards in Britain and America. I regret to say that in my opinion the question bristles with so many difficulties that anyone who tries to be fair to both sides is almost certain to incur the displeasure of both. Any simple answer is, I believe, utterly misleading. Is a school to be criticized for not achieving what it is not trying to do? Are English schools to be condemned as being of a low standard because they make very little direct provision for civic training or for instrumental music? Are American schools to be condemned for making places for such things in their curricula? Your answer must depend on your philosophy of education, which in turn will depend on your philosophy of life.

I have implied that American pupils will normally start algebra at the age of fifteen or sixteen and start geometry at sixteen or seventeen. This seems late to us. On the other hand, pupils with IQ's in the 90's will do this—of course they will be among the last to start, perhaps not until they are eighteen. The abler pupils make very rapid progress when they do start and after eighteen months of algebra will reach G.C.E. ordinary level. In the same way, after a year's geometry they will reach G.C.E. ordinary level. The best pupils will reach this standard at about sixteen and a half or seventeen, which is not far behind our best pupils, and they will have spent a good deal of time at studies which our best pupils have neglected. Thereafter, they proceed along less specialized lines than most of our sixth form pupils, so naturally they do not reach the same level. But is the result any less desirable educationally? The battle of overspecialization rages perpetually in our educational journals. The best answer I can give is that scholarship-level work, or even second-year sixth-form work in mathematics is unknown in American high schools, but I am not satisfied that the result is educationally undesirable anywhere, and I am certain that it is appropriate in the American situation. It is essential to remember that what is done in a school is a reflection of the philosophy of education which prompts the school. Merely because schools are for young people, use blackboards and chalk, and so on, we must not assume that they are prompted by the same philosophies whether they are in England, Wales, Scotland, France, or the United States. They are not, and if they are not, qualitative comparisons become impossible. I prefer an orange to a grapefruit, but even if you agree with me, does that prove that my wife, who doesn't, is wrong?

I have indicated roughly the extent of the various courses offered in the high schools. Perhaps a little more detail of the content of the courses would be of interest. In arithmetic there is a good deal of concern with "consumer arithmetic," by which is meant the elementary arithmetic of buying and selling, meter reading, insurance, banking, installment buying, and so on. There is also some study of the arithmetic of taxation, both from

the national and from the individual point of view. This part of the work is considerably more extensive than anything comparable found in our schools. Under arithmetic is also included elementary work with formulae and the mensuration of common areas and solids, including the circle, cylinder, and prism, and some practical geometry and trigonometry. This course normally finishes when the pupil is fourteen or fifteen, but may continue until the pupil leaves school.

The algebra course is a two-year course, but most pupils take only the first year and a half. This year and a half includes manipulation of literal expressions, directed numbers, linear simple and simultaneous equations, quadratic equations with one unknown, exponents and radicals, functions and graphs, ratio and proportion—in fact, very much the algebra of the old type G.C.E. ordinary papers. Complexity of manipulation, however, is considerably greater than we have practiced for many years. The last half year, usually taken after an interval spent studying formal geometry, includes irrational equations, imaginary and complex numbers, the binomial theorem, permutations and combinations, progressions, and the theory of equations—a formidable list indeed for half a year's work. In practice some selection of material occurs, and the emphasis appears to be on elementary manipulation and application rather than on theoretical development.

Geometry courses, again, match our G.C.E. ordinary courses in content, with perhaps a little more emphasis on the postulates and axioms and inequalities, and a little less emphasis on circles and similar figures than we are accustomed to. Solid geometry is first developed logically from appropriate postulates and axioms, and then a good deal of work is done in calculating lengths, angles, areas, and volumes, especially in connection with prisms, cylinders, cones, and spheres.

The trigonometry courses include solution of right and oblique triangles, functions of angles of any size, trigonometrical equations and identities. The field is about that of the N.U.J.M.B. A22 Mathematics, with rather more emphasis on numerical calculations.

The course in analysis is very wide, and a selection is made from the following range of topics: functions and graphs including simple discontinuities; differentiation and integration (as far as inverse trigonometric functions and exponential and logarithmic functions, but excluding integration other than by very simple substitutions); rectangular co-ordinates (very elementary work with central conics, referred to their centers, and the parabola); polar co-ordinates; solution of equations; definite integrals. The ground covered is very much less than that of our G.C.E. Advanced Mathematics.

I have not yet referred to the most interesting mathematics course I met with during my American visit. When I arrived in Denver, I was asked to take an "A" course in geometry, and in all innocence I said that I would. Since this was to be the pupils' first contact with formal geometry, I thought that I could manage it without undue strain. Judge my horror when I was presented with a textbook bearing the title *Clear Thinking* and found that "Geometry A" really meant a course in elementary applied logic, using geometric postulates, axioms, and theorems as a convenient and easy field, but ranging far and wide in subject matter!

No doubt you all remember the efforts of the Association for the Reform of the Teaching of Geometry, the parent of our Association, and also the first report on "The Teaching of Geometry" published by our Association. To overcome the pertinent objection that, in view of the discoveries of experimental psychology, formal geometry could be considered of very little use to the majority for mind-training or anything else unless its data and their relation to the external world were well understood, the suggestion was made that the teaching of geometry should

fall into three stages: Stage A, Experimental; Stage B, Deductive; Stage C, Systematizing. We have tried to make formal geometry grow out of an exploration of space, so that even if our pupils do not train their minds, they will at least learn some practical geometry. My "Geometry A" course tackled the same problem from a different point of view. Some transfer of training can take place if teaching is directed to that end—that is, if care is taken to emphasize, in this case, the reasoning processes used, and if examples of the application of these processes in a wide variety of fields is deliberately sought. Remember that this geometry course is for pupils of seventeen years. who have already reached G.C.E. ordinary level in algebra, are of fair intelligence, and have voluntarily agreed to take this particular course.

I think you must agree that for general education and culture this course is quite as sound as the one with which we are more familiar. Later, I was told that until a teacher has spent at least three years at this course, building up a suitable body of examples, it is really hard work to teach. I certainly found that it required a lot more thought and preparation than any other course at any level that I have taken. I will also say that I got more satisfaction from this course in the end than from any other course, except possibly my real love—analysis with good scholarship pupils.

The products of the course were listed by students as being: making and recognizing generalizations; avoiding overgeneralizations; understanding the need for generalizations; applying generalizazations; use of indirect reasoning; recognizing faulty forms of thinking, e.g., inadequate data, faulty analogies, misusing converses and inverses, thinking in circles, rationalizing, word reasons, mistaking sequences for causes, begging the question; developing the habit of thinking before speaking; saying what is meant; noting double meanings; use of definitions; following directions precisely or with judgment; recognizing implied assumptions; facing facts; etc. An imposing list! How many of our sixth form pupils get that or even a small part of it from their study of mathematics? I suppose I should add that American teachers are at least as conservative as English teachers, and that I was asked to take the course partly because it was less popular than a more traditional alternative course. However, I repeat, I enjoyed the class and believe that, in the field of teaching, it was one of my most valuable experiences.

On the whole I found that American teaching methods and courses tended to be more formal than our own. Teacher-pupil relations were rather freer than ours, but not freer than are found with individual teachers in most English schools. All the staff, all the pupils I taught, and any others that I came into contact with greeted me every day when we met for the first time. True, the greeting was the standard American, "Hi!," but I missed it very much when it was replaced by the traditional English dour silence. Pupil-teacher relationships were friendly, but there was an innate attitude of respect. Subject matter, on the other hand, was very formal. Lip service was paid to applications of mathematical methods, but the applications were few and despised or feared. The terror of any algebra class was the story problem; the textbooks I used had very few problems, and at the higher levels of study the lack made itself felt. Several pupils in my analysis class told me that their difficulties were largely due to lack of experience in and consequent fear of work of the problem type.

The same formality extended to "Geometry B," the traditional geometry course, which was rather more formal than our Stage B and began with a list of axioms, postulates, etc., to be learnt. I also found it rather difficult to discuss congruence with people who had little idea how to measure an angle and who felt aggrieved if they were asked to verify the

results of reasoning by drawing and measurement. Faith in reasoning is good, but a realization of the frailty of human reasoning is scientific! The solid geometry text I used had a number of very beautiful pictures of airplanes, bridges, and so on, with hints that solid geometry was needed in their construction, but I searched the book in vain for any clue indicating just what was used. I feel that our best textbooks might not have the pictures, but that they would certainly have some examples of applications. It is only fair to add that, although none of the books I used were old, much more satisfactory texts are now appearing.

Teaching techniques seemed to me to be very much the same as in England. I met one or two ideas which were new to me, but which I felt I might easily have met here. One was the institution of a class president, vice-president, and secretary, fulfilling many of the functions of our form-captains or monitors, but also accepting responsibility for checking attendances, calling the class to order, organizing recitations, and so on. The matter of recitations I found interesting. For homework a set of examples would be set, and preparation would often be tested by calling on pupils to do selected examples on the board and to defend their work against criticism by other pupils. The selection of pupils for this purpose and the conduct of the cross-examination was in the hands of the class president. Of course, the teacher added his criticism where necessary, but the pupils usually spotted all the points which called for criticism. Naturally, it was no novelty to see pupils working at the board, but to use it as a systematic method of checking homework in geometry or analysis was novel. I found the method effective. Blackboards were large, and a common practice was to get four or five pupils to prepare their figures and particular enunciations simultaneously at the blackboard, and then go through their proofs in order.

Generally, I think that the standard of teaching in America is of the same order as in Britain, but the tendency for good teachers and highly qualified teachers to leave the schools is greater than it is here. Elevation to administrative heavens is the ambition of the good teacher, and a fair number make the grade. Nevertheless, I met many skillful and capable teachers actually teaching. A further important point is that in many areas all teachers at all levels must be graduates. While admitting that a first degree in an American university is not quite the same thing as a first degree here, yet a university degree of the more generalized type popular there certainly has great value in equipping a teacher.

The conditions under which teachers have to work vary widely—the best no better than our best, the worst no better than our worst, the average, little different from our average. I think schools are less well equipped for mathematics teaching than ours, perhaps because the teaching is more formal. Visual aids are available fairly freely, and I experimented with a number of filmstrips borrowed from a common library, but I found, as with English strips, that they seldom did much that I could not do with equal facility on the blackboard and with less waste of time and general disturbance to the class.

A point of some interest is the way in which some sort of uniformity between schools is achieved in the absence of anything resembling our General Certificate of Education examinations, for a broad uniformity certainly exists. There are two influences making for uniformity. The first is the use of standardized attainment tests. These tests exist in most subjects and at many levels—well over one hundred such tests were in use in Denver—and it is a common practice for a teacher to subject his pupils to one of these tests at the end of each half-year. He does this partly in self-defense so that if there is any criticism of his teaching he can point to

tests results, and partly for his own satisfaction so that he may have a check on his own work. I must admit I found the tests quite helpful, as providing a simple rod to determine the progress of my pupils. While I do not think that any teacher teaches for these tests as we teach for G.C.E., yet their existence in the background is a kind of framework for the courses.

A second influence for uniformity is provided by the universities. High school diplomas are accepted by the universities as evidence of fitness for entrance provided the high school fulfills certain conditions. These conditions include accommodation and qualifications of the staff. The universities also demand certain units of different subjects according to the particular school it is desired to enter, and for these units a certain minimum range of subject matter is expected. Naturally, the textbooks published are in line with these requirements, and so in spite of the absence of G.C.E. there is very nearly as much uniformity between schools in America as there is in this country.

To summarize my impressions, mathematics teaching in the United States is much more like our own than I had expected. Published research is not widely used, and, again contrary to expectations, teaching is rather more formal than in this country. On the other hand, course arrangements are much more flexible, and more students pursue their studies of mathematics to a reasonable level than happens here. Also, as a consequence of arrangements made to make the best use of the varied courses and of the close atmosphere of co-operation between teachers and pupils, studies are generally pursued with a good deal of vitality. However, under existing conditions the best pupils do not proceed as far as our best pupils with their studies. In some ways the American system has the advantage, in some ways ours. I have been happy in both.

What's new?

BOOKS

SECONDARY

Algebra and Its Use, "Book I," Ethel L. Grove, Anne M. Mullikin, Ewart L. Grove, New York, American Book Company, 1956. Cloth, vi+454 pp., $3.20.

Elementary Algebra, William G. Shute, William E. Kline, William W. Shirk, LeRoy M. Willson, New York, American Book Company, 1956. Cloth, viii+488 pp., $3.20.

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Analytic Geometry and Calculus, Thurman S. Peterson, New York, Harper and Brothers, 1955. Cloth, ix+456 pp., $5.50.

Famous Problems and Other Monographs, rev. ed., F. Klein, W. F. Sheppard, P. A. MacMahon, L. J. Mordell, New York, Chelsea Publishing Company, 1955. Cloth xi+319 pp., $3.25.

Mathematics of Finance, second ed., Paul M. Hummel and Charles L. Seebeck, Jr., New York, McGraw-Hill Book Company, Inc., 1956. Cloth, xi+372 pp., $4.75.

Methods in Numerical Analysis, Kaj L. Nielsen, New York, Macmillan Company, 1956. Cloth, xiii+382 pp., $6.90.

Reelle Projektive Geometrie Der Ebene, H. S. M. Coxeter, München, Germany, Verlag Von R. Oldenbourg, 1955. 190 pp., DM, 18,60 Pappb.

FILMSTRIPS

"Man and Measures Series": *Early Counting*, *Early Measuring*, *Early Time Telling*, and *Geometric Figures*. The Filmstrip House, 15 West 46th Street, New York 36, New York. Series of 4-color filmstrips geared to junior and senior high school level, written and illustrated by Agnes Herbert, edited by Reid Irving. $6.00 each, or $20.00 for set.

MODEL

Inverse-Squares Illustration (*No. 3535*), W. M. Welch Scientific Company, 1515 Sedgwick Street, Chicago 10, Illinois. Plastic and metal model showing law of inverse squares, $10.00.

Three algebraic questions connected with Pythagoras' theorem

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The first part of a two-part article with this title appeared in the December issue of The Mathematics Student Journal *(vol. 2, no. 4, December 1955); the concluding second part appears in the April issue of that journal. The present paper proposes to discuss these three questions in a broader mathematical context and thus supply additional material to teachers who work through the* Student Journal *articles with their pupils.*

Introduction

If the measure of the hypotenuse of a right triangle is h and the measures of its other sides are x and y (all measures being in the same units), then

$$x^2+y^2=h^2.$$

We regard this as the algebraic version of Pythagoras' theorem.

With the understanding that *natural* numbers are positive whole numbers, the three algebraic questions posed in *The Mathematics Student Journal* for December 1955 run as follows:

Question One: Are there three natural numbers x, y, h such that

(1) $$x^2+y^2=h^2?$$

Question Two: Are there four natural numbers x, y, u, v such that

(2) $$x^2+y^2=u^4 \quad \text{and} \quad y-x=v^2?$$

Question Three: Are there four natural numbers x, y, u, v such that

(3) $$x^2+y^2=u^4 \quad \text{and} \quad x+y=v^2?$$

Geometric versions of these questions make evident their connection with Pythagoras' theorem. Thus *Question One* asks: Is there a right triangle whose sides have measures that are *natural* numbers? *Question Two* asks: Is there a right triangle whose sides have measures that are *natural* numbers, whose hypotenuse measure is itself a square (i.e., there is a natural number u such that $h=u^2$), and whose other two sides have measures whose *difference* is a square? And *Question Three* asks: Is there a right triangle whose sides have measures that are *natural* numbers, whose hypotenuse measure is itself a square, and whose other two sides have measures whose *sum* is a square?

Now the answer is "yes" for each of *Questions One, Two,* and *Three.* Evidence justifying an affirmative answer to *Question One* is the familiar Pythagorean triple 3, 4, 5; which is to say, the numbers

(4) $$x=3, \quad y=4, \quad h=5$$

satisfy equation (1). Moreover, it is easy to guess many other solutions of (1) in natural numbers. Evidence justifying an affirmative answer to *Question Two* is

harder to come by. After some guesswork and experimenting, however, it appears that

(5) $x=119, \quad y=120, \quad u=13, \quad v=1$

is a solution of (2) in natural numbers. (The right triangle corresponding to solution (5) has for its sides $x=119$, $y=120$, $h=169$; note that these three numbers satisfy (1), that h is itself a square ($h=169=13^2$), and that the difference $y-x$ is also a square ($y-x=120-119=1=1^2$).)

Evidence justifying an affirmative answer to *Question Three* is provided by the fact that the natural numbers

(6)
$$\begin{aligned} x &= 4565486027761, \\ y &= 1061652293520, \\ u &= 2165017, \\ v &= 2372159, \end{aligned}$$

satisfy (3). (The right triangle corresponding to solution (6) has for its shorter sides the numbers x and y given in (6) and for its hypotenuse the number $h=4687298610289$; moreover, this is the "smallest" right triangle providing evidence for *Question Three*.)

The difference between *Question Three* and *Questions One* and *Two* is that it is impossible to obtain the numbers (6) by guess, by accident, or by experiment. That these numbers do satisfy (3) can be verified after some tedious arithmetic; however, what is of interest now is *how* such numbers (6) can ever be discovered at all. The second concluding part of the *Student Journal* article (the part appearing in the April issue) shows how the numbers (6) can actually be obtained using just the simplest tools of elementary algebra.

The above is a concise summary of the content of the two-part *Student Journal* article that is under consideration. Now let us notice briefly the main rhetorical and pedagogical devices employed in the article. *Questions One, Two,* and *Three* above are formulated just as they were in the *Student Journal*, viz., as simple decision (or "yes-no") problems.[1] That formulation was deliberate on our part. We feel the decision problem has some special advantages with high-school pupils; it seems natural, its answer (either "yes" or "no") is no harder to understand than the question, it underscores the necessity for conclusive evidence (exhibit numbers that work or show cause why no such numbers can be found), and it induces intelligent guessing or experimentation.

With each *Question*, numbers were given which the pupil could *recognize* as evidence on the *Question* and *use* to answer the *Question*. (The numbers given the pupils were, in each case, the numbers given above, viz., (4), (5), (6).) Here again, there was deliberate emphasis on the (more passive) recognition of evidence as against the (more active) finding of evidence. No suggestion was given that the finding of evidence could involve systematic mathematical thinking.

The ordering of the *Questions* was also deliberate. Since all the *Questions* see the same use of evidence, *Question One*—arithmetically the easiest—was put first to sharpen the pupil's sense of the relation between evidence and answer. With *Questions Two* and *Three* the problem of recognizing evidence involves arithmetic of increasing complexity. But the subtler aim in the chosen ordering of the *Questions* was to develop within the pupil a feeling that at the heart of each *Question* is the problem of *finding* relevant evidence, to convince him that guessing and experiments are of limited usefulness, and to arouse in him a need to know how such evidence can be found *systematically*. The direct consequence of this need should be a strong drive within the pupil to work through the second part of the paper and see how the solution (6) of (3) can actually be obtained. A further consequence can be

[1] We have in mind here the following three-way classification of (mathematical) problems: problems of demonstration ["Prove that *A* is *B*."]; problems of decision ["Decide whether *A* is *B*."]; and problems of selection ["Find out which *A* is *B*."].

hoped for: curiosity on the pupil's part to see a systematic solution of equation (1) in natural numbers.

Elaboration of Question One

The following elaboration of *Question One* is intended to help a teacher who wants to show a pupil how equation (1) may be solved systematically in natural numbers. While we imagine such a teacher would want to work through the elaboration below, it would be our suggestion that his point of departure *with the pupil* be the article "Pythagorean Triplets" in the December, 1954 issue of *The Mathematics Student Journal*.

We begin by reformulating *Question One: Question* I: Find all the triples (x, y, h) such that x, y, and h are natural numbers and

$$x^2+y^2=h^2. \tag{1}$$

The difference between *Question* I and *Question One* should be noted. *Question* I is not a decision problem, but a selection problem; from all the triples (x, y, h) of natural numbers, select those which satisfy condition (1). Pupils who on their own manage the passage from *Question One* to *Question* I may rightly be regarded as having mathematical insight into *Question One*.

Question I is of considerable antiquity. The solution of it given below was known to Diophantus in the third century A.D.[2] Our treatment of this *Question*, while terse, is complete. We begin with some familiar definitions.

A *natural number* is any one of $1, 2, 3, 4, 5, \cdots, n, n+1, \cdots$; if n and m are natural numbers, so also is their sum $n+m$ and their product mn. The difference $n-m$ of two natural numbers is a natural number just in case $n>m$. A natural number n is *even* in case $n=2a$ for some natural number a. A natural number n is *odd* in case $n=2a-1$ for some natural number a. Two natural numbers are of the *same parity* in case both are even or both are odd; they are of *different parity* in case one of them is even and the other odd. Now it is easy to establish the following:

Lemma 1. The natural numbers n and n^2 are of the same parity.

Proof. Suppose first that n is even. Then n^2 is also even; for if $n=2a$, then $n^2=(2a)^2=4a^2=2(2a^2)$. Again, suppose n is odd. Then n^2 is also odd; for if $n=2a-1$, then

$$n^2=(2a-1)^2=4a^2-4a+1$$
$$=2(2a^2-2a+1)-1.$$

Conversely, if n^2 is even, then n must be even; for if n were odd, n^2 would also be odd. And if n^2 is odd, then n must be odd; for if n were even, n^2 would be even too.

Lemma 2. If n and m are natural numbers such that $n>m$, then: n and m are of the same parity if and only if $n+m$ and $n-m$ are even.

Proof. Suppose $n>m$. Both $n+m$ and $n-m$ are then clearly natural numbers. The proof now falls into two parts. In this first part, we *assume* n and m are of the same parity and *show* that $n+m$ and $n-m$ must then be even. Taking n and m to be odd, we know there are natural numbers a and b such that $a>b$ and $n=2a-1$ and $m=2b-1$; but then

$$n+m=(2a-1)+(2b-1)=2(a+b-1)$$

and

$$n-m=(2a-1)-(2b-1)=2(a-b),$$

whence both $n+m$ and $n-m$ are evidently even. Taking n and m to be even, we know there are natural numbers a and b such that $a>b$ and $n=2a$ and $m=2b$; but then $n+m=2(a+b)$ and $n-m=2(a-b)$, whence again $n+m$ and $n-m$ are even. Turning to the second part of the proof, we

[2] For further historical details on *Question* I, see: Dickson, *History of the Theory of Numbers* (Washington: Carnegie Institution of Washington, 1920), vol. 2, pp. 165–171. For interesting computational details, see Hart, "Pythagorean Numbers," The Mathematics Teacher, vol. XLVII (January 1954), pp. 16–21.

assume $n+m$ and $n-m$ are even and *argue* that n and m must then be of the same parity. But if n and m were of different parity, one would be even and the other odd; and in either event both $n+m$ and $n-m$ would be odd. This last being contrary to assumption, if follows n and m cannot be of different parity.

A natural number k *divides* a natural number n in case $n=ka$ for some natural number a. Other ways to say "k divides n" are "*n is divisible by k*," "*k is a divisor of n*," and "*k is a factor of n*." Thus a natural number is even in case it is divisible by 2; otherwise, it is odd.

Lemma 3. If n is a natural number, then: n^2 is even if and only if 4 is a factor of n^2.

Proof. Suppose n is a natural number. Now by *Lemma 1* we know that n^2 is even if and only if n is even. Since n is even if and only if there is a natural number a such that $n=2a$, it follows that n^2 is even if and only if $n^2=4a^2$ for some natural number a. But $n^2=4a^2$ if and only if 4 is a factor of n^2.

Lemma 4. If n and m are natural numbers such that $n>m$ and the natural number k divides both $n+m$ and $n-m$, then whatever natural number divides k also divides both $2n$ and $2m$.

Proof. Since k is supposed to divide both $n+m$ and $n-m$, there must be natural numbers a and b such that $a>b$ and $n+m=ka$ and $n-m=kb$. Hence $2n=k(a+b)$ and $2m=k(a-b)$. From these last two equations it appears that whatever divides k must also divide $2n$ and $2m$.

A natural number n is *prime* in case $n>1$ and the only divisors of n are 1 and n.[3] Two natural numbers n and m have a divisor k in common in case $n=ka$ and $m=kb$ for some natural numbers a and b; if n and m have a divisor k in common, k is a *common divisor* of n and m. What it means to say that k is a common divisor of three or more natural numbers now follows by analogy. Finally, two or more natural numbers are *relatively prime* in case their only common divisor is 1.

Now let us revert to *Question* I. Suppose S is the set of all triples (x, y, h) such that x, y, and h are natural numbers and $x^2+y^2=h^2$. Our problem is to specify each member of S, or to show how all the triples in S can be generated. First, we note two obvious facts.

Theorem 1. If (x, y, h) is a triple in S, then so is the triple (y, x, h).

Proof. Suppose (x, y, h) is in S. Then $x^2+y^2=h^2$. But $x^2+y^2=y^2+x^2$. Hence $y^2+x^2=h^2$, and so (y, x, h) also belongs to S.

Theorem 2. If (x, y, h) is a triple in S, then so is the triple (kx, ky, kh) for any natural number k.

Proof. Suppose (x, y, h) is in S. Then $x^2+y^2=h^2$. Multiplying this equation, through by k^2 (the square of the natural number k), we obtain $k^2x^2+k^2y^2=k^2h^2$, i.e., $(kx^2)+(ky)^2=(kh)^2$. From this last it follows that (kx, ky, kh) is also in S.

Theorem 2 tells us that from a given triple in S indefinitely many other triples in S can be generated. E.g., the triple (3, 4, 5) belongs to S, hence so does each of the following:[4]

$$(6, 8, 10)=(2\cdot 3, 2\cdot 4, 2\cdot 5),$$
$$(9, 12, 15)=(3\cdot 3, 3\cdot 4, 3\cdot 5),$$
$$(12, 16, 20)=(4\cdot 3, 4\cdot 4, 4\cdot 5),$$
$$\text{etc.}$$

Similarly, the triple (5, 12, 13) belong to S, hence so does each of the following:

$$(10, 24, 26)=(2\cdot 5, 2\cdot 12, 2\cdot 13),$$
$$(15, 16, 39)=(3\cdot 5, 3\cdot 12, 3\cdot 13),$$
$$(20, 48, 52)=(4\cdot 5, 4\cdot 12, 4\cdot 13),$$
$$\text{etc.}$$

[3] Students often ask why this definition includes the condition that $n>1$. The answer: if that condition were removed and hence 1 admitted as a prime, then, e.g., each of $2\cdot 3\cdot 5$, $1\cdot 2\cdot 3\cdot 5$, $1\cdot 1\cdot 2\cdot 3\cdot 5$, etc., would be *different* factorizations of 30 into primes, and the so-called Fundamental Theorem of Arithmetic would not be true.

[4] We say $(a, b, c)=(p, q, r)$ in case $a=p$ and $b=q$ and $c=r$.

Clearly now, the key triples in S are those triples (x, y, h) whose co-ordinates x, y, h have *no factors in common*. Let us give such triples a special name.

Definition 1. We say that a triple (a, b, c) is a *principal triple of* S in case (a, b, c) belongs to S and the three numbers a, b, c are relatively prime.

E.g., (3, 4, 5), (5, 12, 13), and (7, 24, 25) are principal triples of S, whereas (4, 8, 10), (15, 36, 39), and (21, 72, 75) are not. If we can identify all the principal triples of S, then (with the help of *Theorem 2*) we can generate all the triples of S and hence answer *Question* I. Let us seek a characterization of the principal triples of S.

Theorem 3. If (a, b, c) is a principal triple of S, then: a and b are relatively prime, a and c are relatively prime, and b and c are relatively prime.

Proof. Suppose (a, b, c) is a principal triple of S. Now suppose further that a and b are *not* relatively prime. Hence, there must be natural numbers k, d, e such that: $k>1$, and $a=kd$, and $b=ke$. But then, since $a^2+b^2=c^2$ and

$$a^2+b^2=k^2d^2+k^2e^2=k^2(d^2+e^2)$$

it must be the case that $c^2=k^2(d^2+e^2)$. Since d^2+e^2 is plainly a natural number, we see that k^2 divides c^2 and hence that k divides c. Because c also has the number k as a factor, each of a, b, c have in common the factor k, where $k>1$. This being impossible (because (a, b, c) is supposed to be a principal triple of S), it cannot be the case that a and b are not relatively prime. Therefore a and b are relatively prime. Exactly similar arguments serve to show that a and c, and b and c, are relatively prime.

Theorem 4. If (a, b, c) is a principal triple of S, then a and b are of different parity.

Proof. Suppose (a, b, c) is a principal triple of S. Clearly a and b cannot both be even, for then they would not be relatively prime—contrary to *Theorem 3*. Suppose, therefore, that both a and b are odd. Then there are natural numbers d and e such that $a=2d-1$ and $b=2e-1$. Since (a, b, c) belongs to S, we know therefore that

$$e^2=a^2+b^2=(2d-1)^2+(2e-1)^2,$$
$$c^2=(4d^2-4d+1)+(4e^2-4e+1),$$
$$c^2=2[2(d^2+e^2-d-e+1)-1].$$

This last equation tells us c^2 is even, but does *not* contain 4 as a factor (since the number $2(d^2+e^2-d-e+1)-1$ is obviously odd). By *Lemma 3*, therefore, c cannot be a natural number. This means that (a, b, c) cannot belong to S, contrary to our initial supposition. Hence, a and b cannot both be odd, and we have established *Theorem 4*.

The four lemmas and the four theorems now at hand enable us to characterize any principal triple of S. This characterization is given in

Theorem 5. The triple (a, b, c) is a principal triple of S if and only if there are natural numbers n and m such that

$$(7) \qquad \begin{aligned} a&=2mn,\\ b&=n^2-m^2,\\ c&=n^2+m^2 \end{aligned}$$

where $n>m$, n and m are relatively prime, and n and m are of different parity.

Before proving this theorem, let us probe it. Will the choice $m=2$, $n=3$ yield a principal triple (a, b, c) of S? The answer should be "yes," because here $n>m$, n and m are relatively prime, and n and m are of different parity. Substituting in (7), we obtain $a=12$, $b=5$, $c=13$. The triple $(a, b, c)=(12, 5, 13)$ is obviously a principal triple of S. Conversely, given the principal triple (24, 7, 25) of S, how can we find numbers n and m, which relate to 24, 7, 25 in the manner of the theorem? Since $24^2+7^2=25^2$, we see that

$$24^2=25^2-7^2=(25-7)(25+7)$$
$$=18\cdot32=4\cdot9\cdot16.$$

By (7) our desired numbers n and m must be such that $a=2mn$, i.e., $4m^2n^2=a^2=4\cdot9\cdot16$. This last suggests the choice

$m=3$, $n=4$—numbers which are such that $n>m$, n and m are relatively prime, and n and m are of different parity. Computing a, b, c, by (7), we obtain $a=2\cdot3\cdot4=24$, $b=16-9=7$, $c=4^2+3^2=25$. Hence the use of $m=3$, $n=4$ in *Theorem 5* yields the principal triple (24, 7, 25).

It is also easy to see that the numbers n and m of *Theorem 5* must satisfy the three conditions imposed on them there. Thus, it must be the case that $n>m$; otherwise, b would not be a natural number. Again, if n and m are not relatively prime, the triple resulting from (7) need not be a principal triple of S: e.g., if $n=6$, $m=3$ we have $a=36$, $b=27$, $c=45$; while (36, 27, 45) belongs to S, it is not a principal triple of S. And again, if n and m are of the same parity, the triple resulting from (7) need not be a principal triple of S: e.g., if $n=3$, $m=1$ we have $a=6$, $b=8$, $c=10$; while (6, 8, 10) belongs to S, it is not a principal triple of S.

Now let us turn to the proof of *Theorem 5*.

Proof of Theorem 5. The proof divides into two parts. The first shows that *if* natural numbers n and m are as specified at the end of *Theorem 5*, *then* the triple (a, b, c) formed from the numbers a, b, c of (7) is indeed a principal triple of S. The second part shows that *if* (a, b, c) is a principal triple of S, *then* there are natural numbers n and m which satisfy (7) and the other specifications of *Theorem 5*.

Part One. Let n and m be natural numbers such that $n>m$, n and m are relatively prime, n and m are of different parity, and (7) holds. Then a and c are clearly natural numbers, and so is b, since $n^2>m^2$ because $n>m$. Furthermore, that (a, b, c) is a triple belonging to S follows at once from (7):

$$\begin{aligned} a^2+b^2 &= (2mn)^2+(n^2-m^2)^2 \\ &= 4m^2n^2+n^4-2m^2n^2+m^4 \\ &= n^4+2m^2n^2+m^4 \\ &= (n^2+m^2)^2 \\ &= c^2. \end{aligned}$$

It remains to be shown that (a, b, c) is a principal triple of S, i.e., the numbers a, b, c are relatively prime. Since n and m are of different parity, the same is true of n^2 and m^2 (by *Lemma 1*). Hence, the numbers b and c are odd (by *Lemma 2*), and thus the numbers a, b, c cannot have an even factor in common. Suppose a, b, c had in common an odd factor greater than 1. Then, in particular, b and c would have this odd factor in common, and therefore so would n^2 and m^2 because

$$2n^2=c+b \quad \text{and} \quad 2m^2=c-b.$$

But if n^2 and m^2 have a common factor, so must n and m—and this last is contrary to our assumption that n and m are relatively prime. Hence the numbers a, b, c are relatively prime and the triple (a, b, c) is a principal triple of S.

Part Two. Here we assume that (a, b, c) is some principal triple of S and must show that then there can be found natural numbers n and m such that $n>m$, n and m are relatively prime, n and m are of different parity, and (7) holds. Suppose (a, b, c) is a principal triple of S. Then by *Theorem 4* we know a and b are of different parity, i.e., one of them is even and the other odd. By *Theorem 1* we lose no generality in *assuming* that *a is even* and *b, odd*. Also c must be odd in view of *Theorem 3*. Now from the fact that $a^2+b^2=c^2$, we see that $c>b$ (since clearly $c^2>b^2$) and that

$$(8) \qquad a^2=c^2-b^2=(c+b)(c-b).$$

Numbers c and b being both odd, we see by *Lemma 2* that $c+b$ and $c-b$ are both even and hence have a factor 2 in common. Moreover, $c+b$ and $c-b$ have no other factor greater than 1 in common; for if they did, this factor would divide both $2b$ and $2c$ (by *Lemma 4*) —and since b and c are relatively prime, the only common factor $2b$ and $2c$ can have is 2. If, therefore, as (8) tells us, the product of $c+b$ and $c-b$ is to be the square of a natural

number, then there must be relatively prime natural numbers n and m such that

$$c+b=2n^2 \quad \text{and} \quad c-b=2m^2$$

and

$$a^2=(c+b)(c-b)=4n^2m^2.$$

From these three equations we readily obtain (7), i.e., we may deduce that $a=2mn$, $b=n^2-m^2$, and $c=n^2+m^2$. Also we see that $n>m$: for $c+b>c-b$, hence $2n^2>2m^2$ and so $n>m$. And finally, it is the case that n and m are of different parity: both b and c being odd, both n^2-m^2 and n^2+m^2 are, hence (by *Lemma 2*) the numbers n^2 and m^2 are of different parity, consequently (by *Lemma 1*) so are n and m. We have finished *Part Two* and, thereby, the proof of *Theorem 5*.

When *Theorems 1*, *2*, and *5* are combined, we obtain the desired characterization of triples in S: *(x, y, h) is a triple of natural numbers such that $x^2+y^2=h^2$ if and only if there are natural numbers k, n, m such that $n>m$, n and m are relatively prime, n and m are of different parity, and either*

$$\begin{aligned} x&=2kmn, & x&=k(n^2-m^2),\\ y&=k(n^2-m^2), \quad \textit{or} & y&=2kmn,\\ h&=k(n^2+m^2), & h&=k(n^2+m^2). \end{aligned}$$

This represents the complete solution of *Question* I.[5]

Remarks on Questions Two and Three

An elaboration of *Questions Two* and *Three* to the same extent as that just given for *Question One* is beyond the scope of this paper, for a reason to be discussed next.

Consider first *Question Two*. Its reformulation would run:

Question II: Find all the quadruples (x, y, u, v) such that x, y, u, v are natural numbers and

$$(2) \qquad x^2+y^2=u^4 \quad \text{and} \quad y-x=v^2.$$

Now it is the case that *Question* II is *equivalent* to the following *Question* II′:

Question II′: Find all the triples (u, v, w) such that u, v, w are natural numbers, $u>v$, and

$$(9) \qquad 2u^4-v^4=w^2.$$

For suppose we had a solution (x, y, u, v) of (2). Then, since $u^4-v^4=x^2+y^2-(y-x)^2=2xy>0$, we know that $u^4>v^4$, and so $u>v$; further, $2u^4-v^4=2x^2+2y^2-(y-x)^2=(x+y)^2$. Agreeing that $x+y=w$, we thus see that *Question* II implies *Question* II′. Conversely, suppose a triple (u, v, w) to be such that (9) holds and $u>v$. It is then the case that $w>v^2$ (for if $u>v$, then $u^4>v^4$; thus $2u^4-v^4>v^4$, hence by (9) we have $w^2>v^4$ and thus $w>v^2$), and further that w and v^2 are of the same parity (for since $w^2+v^4=2u^4$, w^2 and v^4 must be of the same parity by *Lemma 2*, hence so must w and v^2 by *Lemma 1*). Taking x and y such that

$$x=\frac{w-v^2}{2} \quad \text{and} \quad y=\frac{w+y^2}{2},$$

we see that x and y are both natural numbers (since by *Lemma 2* $w-v^2$ and $w+v^2$ are both even) and that

$$x^2+y^2=\tfrac{1}{4}[2w^2+2v^4]=\tfrac{1}{2}[2u^4]=u^4,$$

$$y-x=\tfrac{1}{2}[w+v^2-(w-v^2)]=\tfrac{1}{2}[2v^2]=v^2.$$

Thus it appears that *Question* II′ implies *Question* II. The two questions are therefore equivalent.

Proceeding in the same vein respecting *Question Three*, we see that its reformulation would be:

Question III: Find all the quadruples (x, y, u, v) such that x, y, u, v are natural numbers and

$$(3) \qquad x^2+y^2=u^4 \quad \text{and} \quad x+y=v^2.$$

[5] A treatment of *Question* I having features in common with the one given here may be found in the book (now unfortunately out of print): Arnold Dresden, *An Invitation to Mathematics* (New York: Henry Holt and Co., 1936). Another useful little book in this connection is: W. Lietzmann, *Der Pythagoreische Lehrsatz* (Leipzig: B. G. Teubner).

This is equivalent to:

Question III′: Find all the triples (u, v, w) such that u, v, w are natural numbers, $v>u$, and

$$2u^4-v^4=w^2. \tag{9}$$

The solutions of *Questions* II′ and III′ both turn on the complete solution of (9) in natural numbers. The treatment of (9) requires rather extensive knowledge of number theory. It is for this reason that we do not give here elaborations of *Questions Two* and *Three* similar to our elaboration of *Question One*.

Instead, we shall present some program notes on the explanation connected with *Question Three* (and given in the *Student Journal* for April 1955) as to *how* to find the numbers (6) that actually satisfy (3). As mentioned in the Introduction, this explanation is elementary; in fact, it requires only the multiplication and factorization of simple algebraic expressions, the solution of linear equations, and the solution of a system of two linear equations. Our remarks center on *Question Three*, but the evident parallel between *Questions Two* and *Three* makes it possible to transfer much of what we say here to *Question Two*. We leave this transference for the most part to the reader.

We shall not repeat the explanation given in the April issue of the *Student Journal*. We assume the reader has worked through that explanation and has it before him for reference now.

Supposing a knowledge of *Theorem 5* at hand, let us undertake to find directly a solution of

$$x^2+y^2=u^4 \quad \text{and} \quad x+y=v^2 \tag{3}$$

in natural numbers x, y, u, v. First we note certain natural conditions on any numbers x, y, u, v that satisfy (3). Thus, e.g., u and v must be such that $v>u$; for $v^4=x^2+2xy+y^2=u^4+2xy$, whence $v^4>u^4$ and so $v>u$. And again, the first three numbers x, y, u must be such that x, y, u^2 is a solution of (1), i.e., a triple belonging to S.

This last remark provides a clue. Recalling *Theorems 1* and *5*, a reasonable first step towards a solution of (3) would be to take $n>m$ and set

$$\begin{aligned} x&=n^2-m^2,\\ y&=2mn \end{aligned} \tag{10}$$

in (3). After this substitution, the first member of the first equation of (3) becomes

$$x^2+y^2=(n^2-m^2)^2+4m^2n^2=(n^2+m^2)^2;$$

whence the first equation of (3) appears as $(n^2+m^2)^2=u^4$, and as $n^2+m^2=u^2$ after simplification. The second equation of (3) becomes simply: $n^2+2mn-m^2=v^2$. Consequently, to find natural numbers x, y, u, v such that (3) holds leads *via* (10) to the new problem: To find natural numbers n, m, u, v such that $n>m$, $v>u$, and

$$n^2+m^2=u^2 \quad \text{and} \quad n^2+2mn-m^2=v^2. \tag{11}$$

If anything, (11) looks harder to solve than (3). But suppose we could find new expressions for u and v in terms of m and n. Such expressions could be substituted into (11), and the resulting equations in m and n might be solvable for m and n as explicit numbers. Let us look into this possibility. Subtracting the first equation of (11) from the second there, we obtain

$$v^2-u^2=2mn-2m^2.$$

Factoring each member of this equation, there results

$$(v-u)(v+u)=2m(n-m). \tag{12}$$

Setting corresponding factors of (12) equal, we have $v-u=2m$ and $v+u=n-m$, whence plainly $2v=n+m$ and $2u=n-3m$. Substituting these values of u and v into (11), we see that both equations yield the same quadratic in m and n, viz.,

$$3n^2+6mn-5m^2=0.$$

Unfortunately, the solutions of the equation $3(n/m)^2+6(m/n)-5=0$ are irrational. Since satisfactory numbers n and m must be such that n/m is rational, this

procedure does not give us useful values for m and n.

However, we should not be hasty in discarding the general idea of the above method. There may be other ways of obtaining new expressions for u and v in terms of m and n besides simply subtracting the two equations. To give the method a fair chance, suppose we multiply the first equation of (11) through by q^2 and the second by p^2, and then subtract the first from the second. The resulting equation

$$p^2v^2-q^2u^2=(p^2-q^2)n^2+2p^2mn-(p^2+q^2)m^2$$

can be written in the form

$$(pv-qu)(pv+qu)=(p^2-q^2)n^2+2p^2mn-(p^2+q^2)m^2. \tag{13}$$

The right member of (13) factors just in case its discriminant $2p^4-q^4$ is a perfect square, i.e., if there is a natural number r such that

$$2p^4-q^4=r^2. \tag{14}$$

(Note the resemblance between (14) and (9).)

The most obvious solution of (14) is $p=1$, $q=1$, $r=1$; this choice converts (13) into (12), and hence corresponds to our initial (unsuccessful) step of simply subtracting the first equation of (11) from the second. But another solution of (14) is $p=13$, $q=1$, $r=239$. With this choice, (13) becomes

$$(13v-u)(13v+u)=(12n-5m)(14n+34m). \tag{15}$$

The obvious course here is to set $13v-u=12n-5m$ and $13v+u=14n+34m$. Together, these equations tell us that $2u=2n+39m$. Using this value of u in the first equation of (11), we find that $4\cdot39n+(39^2-4)m=0$. But this result is also useless; it tells us that these particular numbers n and m are of opposite sign, and hence by (10) that $y=2mn$ is negative, which again is impossible.

Now comes a trick. Noting that

$$(12n-5m)(14n+34m)=\frac{7}{6}(12n-5m)\cdot\frac{6}{7}(14n+34m),$$

that is,

$$(12n-5m)(14+34m)=\left(14n-\frac{35}{6}m\right)\left(12n+\frac{204}{7}m\right),$$

let us rewrite (15) as

$$(13v-u)(13v+u)=\left(14n-\frac{35}{6}m\right)\left(12n+\frac{204}{7}m\right) \tag{16}$$

and set

$$\begin{aligned}13v-u&=14n-\frac{35}{6}m,\\13v+u&=14n+\frac{204}{7}m.\end{aligned} \tag{17}$$

Solving the two equations of (17) simultaneously, we find the following new expressions for u and v:

$$\begin{aligned}2u&=-2n+\frac{1469}{42}m,\\26v&=26n+\frac{979}{42}m.\end{aligned} \tag{18}$$

The first equation comes to

$$84u=-(84n-1469m),$$

whence

$$84^2u^2=84^2n^2-2\cdot84\cdot1469nm+1469^2m^2.$$

From the first equation of (11),

$$84^2u^2=84^2n^2+84^2b^2.$$

Equating these two expressions for 84^2u^2, we find:

$$2\cdot84\cdot1469n=(1469^2-84^2)m. \tag{19}$$

The obvious solutions of (19) are

$$\begin{aligned}n&=1469^2-84^2=2150905,\\m&=2\cdot84\cdot1469=493584.\end{aligned} \tag{20}$$

Using the values (20) in (18) to determine u and v, and in (10) to determine x and y, we obtain precisely the numbers (6) originally announced as a solution of (3).

Now for some comments on the derivation just given of the numbers (6).

First. Our substitution (10) could just as well have been

$$x=2mn$$
$$y=n^2-m^2;$$

no essential change in our derivation would result.

Second. Suppose we were to apply the method of this derivation to *Question Two.* In that case considerations parallel to those above based on (15) actually produce values of n and m (and so of u, v, x and y) which satisfy *Question Two.* However, these values are much larger than those proposed in (5). (This last observation emphasizes that our derivation of the numbers (6) throws no light on why they provide the "smallest" right triangle giving evidence on *Question Three.* This fact must be established independently; it is beyond the scope of the present paper to do so.)

Third. Our solution $p=13$, $q=1$, $r=239$ of the discriminant equation (14) is actually a solution $u=13$, $v=1$, $w=239$ of equation (9) of *Question* II′, since $u>v$. This solution provides the numbers (5) originally seen to satisfy (2).

Finally, some comments on the *Student Journal* derivation of the numbers (6) that satisfy (3).

First. Instead of using the substitution (10) based on *Theorem 5*, the *Student Journal* derivation began straightway with the *identity.*

$$[A^2-B^2]+[2AB]^2=[A^2+B^2]^2.$$

(We recognize the "A" of this identity as our "n," and the "B" as our "m.")

Second. Instead of dealing throughout with the two variables "A" and "B," the *Student Journal* reduced to one variable by substituting into the identity as follows:

$$A=t+5,$$
$$B=12$$

In addition to making the problem a "one-variable" problem, this substitution made implicit and obvious the use of the solution $p=13$, $q=1$, $r=239$ we employed for the discriminant (14); for the substitution gives in place of our system (11) the system

$$t^2+10t+169=u^2,$$
$$t^2+34t+1=v^2.$$

(These are equations (5) and (6) of the *Student Journal* derivation.)

Third. In the present paper we examined the blind alley that began with equation (15). Had a similar examination been interpolated into the *Student Journal,* it would have gone as follows: Beginning with the equation

$$(13v-u)(13v+u)=12t(14t+478),$$

we set

$$13v-u=12t \quad \text{and} \quad 13v+u=14t+478$$

and obtain therefrom

$$u=t+239.$$

Substituting this value of u into the right member of the first of the two equations displayed at the end of the previous paragraph, there results the quadratic

$$t^2+478t+239^2=t^2+10t+169$$

whose solution, t, is a negative number. This value of t is unsuitable because t must be greater than 7 if the x and y produced by (4) of the *Student Journal* are to be positive numbers.

Fourth. The equations (16) and (17) that we obtain after our trick of multiplying by 1 in the form $7/6\cdot 6/7$ have their counterparts in equations (7) and (8) of the *Student Journal* derivation.

Fifth. The *Student Journal's* use of "$t+5$" and "12" in place of "n" and "m" leads to the nuisance of converting from a fractional to an integral solution. This is the penalty paid for avoiding the use of two variables, "n" and "m."

Graphing in elementary algebra

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Graphing in high school and college algebra has been too narrowly interpreted. This paper offers some good suggestions for broadening the topic to include other fundamental mathematical ideas.

MOST MATHEMATICS TEACHERS are constantly seeking varied and challenging exercises which will provide their students with practice, insight, and an understanding of fundamental concepts. In this paper we consider the topic of graphing in elementary algebra from the point of view of such teachers.

One of the basic concepts to be learned in the graphing of equations is that the graph of an equation is the set of points whose co-ordinates satisfy the equation. We shall refer to this as the "locus concept" of graphing. Although students need a lot of practice with this concept in order to achieve a thorough understanding, the assignment of long lists of equations to be graphed does not necessarily provide this practice. The students usually discover mechanical procedures for plotting graphs or are shown such procedures by adults or advanced students who are trying to be helpful. While recognizing that discovery is an important aspect of learning mathematics and that mechanical processes are most useful in routine applications of mathematical ideas, we feel that learning stops when routine begins. Our objection to mechanical procedures for graphing equations is based upon the fact that the student seldom needs to return to the fundamental locus concept in order to draw the desired graph. From our point of view a series of exercises is needed which will compel the student to make direct applications of the locus concept in a non-mechanical fashion.[1]

GRAPHING ON THE NUMBER LINE

In an earlier paper[2] we considered exercises in which students described sets of points or loci on a number line. In effect, the students were graphing equations, inequalities, and other assertions involving a single variable in a one-dimensional space. Since graphing on a number line is based upon concepts which may be readily extended to provide a basis for graphing in a two-dimensional space, we shall reconsider very briefly the concept of graphing on a number line.

Consider the number line as an ordered set of points in one-to-one correspondence with the set of real numbers. The one-to-one correspondence implies that each point on the number line is associated with a unique real number and that each real number is associated with a unique point. The real number is called the *co-ordinate* of

[1] The ideas and procedures in this article are based upon the experiences of the authors as members (1951–54) of the University of Illinois Committee on Secondary School Mathematics (UICSSM). The co-operation of the other members of the committee (Messrs. Babb, Nichols, Page, Pingry, and Stegmeir) in these experiences is hereby acknowledged. The point of view of the committee is evidenced in the recent mimeographed revision of its ninth-year textbook entitled *High School Mathematics—First Course* (Urbana, Illinois: The Committee, University High School, 1954).

[2] Max Beberman and Bruce E. Meserve, "The Concept of a Literal Number Symbol," THE MATHEMATICS TEACHER, XLVIII (April 1955), pp. 198–202.

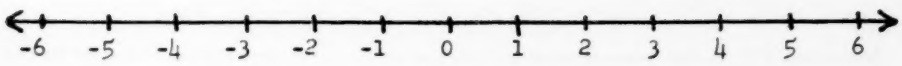


Figure 1

its associated point; the point is called the *graph* of its associated real number. The correspondence between the points and their co-ordinates is completely determined by the choice of the origin and the unit point on the number line, i.e., by the selection of the points with co-ordinates 0 and 1, respectively. It is customary to consider the number line in a horizontal position and to select the unit point on the right of the origin. Then, if point A is on the right of point B, the co-ordinate of A is greater than the co-ordinate of B. A portion of the number line is shown in Figure 1.

We may now state relationships between points on the number line by stating relationships between their co-ordinates. Since the set of points on the number line is in one-to-one correspondence with the set of real numbers, we use the variable "x" in place of a name for an unspecified member of the set of real numbers. That is, we use x as the co-ordinate of an unspecified member of the set of points on the number line. Then any statement concerning points on the number line may be stated in terms of x; any statement about an unspecified real number x may be interpreted or visualized on the number line. For example, the statement "$x^2>4$" is true (or "satisfied") when x is any member of the set composed of all real numbers which either exceed 2 or are less than -2. It is false (or "not satisfied") when x is any member of the set of all real numbers which do not exceed 2 and are not less than -2. Similarly, the statement "$x+3=9$" is true when x is any member of the set composed of the single real number 6; it is false when x is any member of the set of all real numbers which are different from 6. The statement "$|x|<0$" is true when x is any member of the *null* set (often called the *empty* set); it is false when x is any member of the set of all real numbers. The statement "$2x=x+x$" is true when x is any member of the set of all real numbers; it is false when x is any member of the null set. In general, we *solve* a statement involving a single unspecified real number x by determining a set of real numbers such that

1. The statement is true when x is any member of the set, and
2. The statement is false when x is any real number which is not a member of the set.

The set of all "values" of x that satisfy a statement about x is called *the solution of the statement.* The solution of a statement is unique. The reader will recognize, of course, that the above two characteristics of the solution of a statement involving x are equivalent to the familiar necessary and sufficient conditions of locus problems. This equivalence provides one basis for the interpretation of algebraic statements in terms of geometric concepts.

We have defined the solution of any statement about x as the corresponding set of real numbers which has the above characteristics. The graphs of these real numbers constitute a set of points on the number line. This set of points can be called the graph of the solution of the statement or, simply, the *graph of the statement.* Then to "graph" a statement about x means to determine a set of points on the number line with the following characteristics:

1. Every point in the set has a co-ordinate which satisfies the statement, and
2. Every point not in the set has a co-ordinate which does not satisfy the statement.

The graphs of the statements given above are shown in Figure 2. Note that the graphs are indicated by heavy lines. In (1) of Figure 2 the "o" indicates a boundary

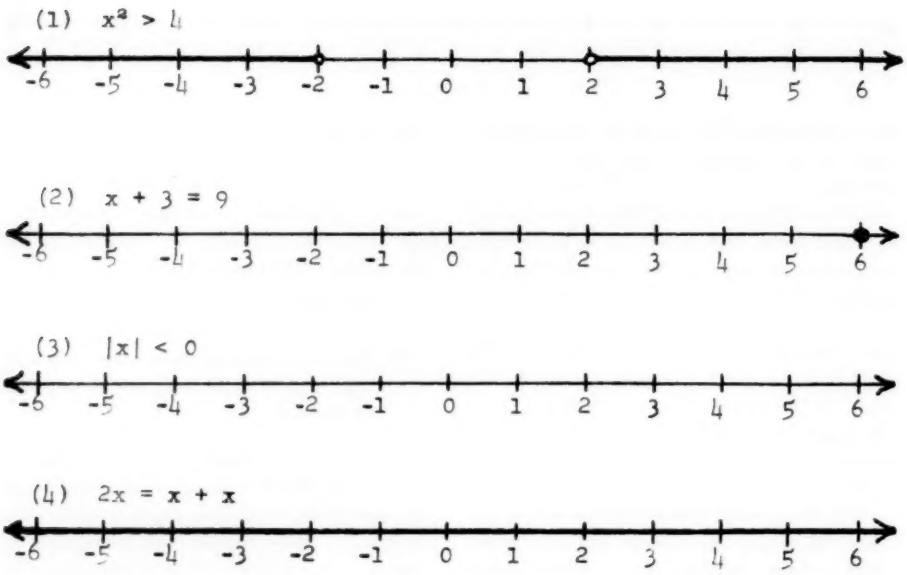


Figure 2

point which is not on the graph; the heavy dot in (2) indicates a point which is on the graph.[3]

Graphing on the Cartesian plane

We now extend the principles encountered in the preceding section to graphing in a two-dimensional space. The familiar Cartesian plane may be considered as a set of points in one-to-one correspondence with the set of all ordered pairs of real numbers. Each point on the plane is associated with a unique ordered pair of real numbers; each ordered pair of real numbers is associated with a unique point on the plane. The numbers in the pair are called the *co-ordinates* of the point; the point is called the *graph* of the ordered pair. The ordered pair of co-ordinates refers to an ordered pair of number lines called the co-ordinate axes. These axes are usually taken with a common origin, with the first co-ordinate referring to a horizontal axis having its unit point on the right of the origin, and with the second co-ordinate referring to a vertical axis having its unit point above the origin. We call such a co-ordinate plane a *Cartesian plane*. The correspondence between the points on the co-ordinate plane and their co-ordinates is then completely determined by the choice of three points—the common origin and the unit points on the two co-ordinate axes.

We may now state relationships among points on the co-ordinate plane by stating relationships among their co-ordinates.

Since the set of points on a co-ordinate plane is in one-to-one correspondence with the set of ordered pairs of real numbers, we use an ordered pair of variables "(x, y)" in place of a name for an unspecified member of the set of ordered pairs of real numbers. Then any statement about a point on the co-ordinate plane may be expressed as a statement about the x-co-ordinate and the y-co-ordinate of the point; any statement about two unspecified real numbers, x and y, may be interpreted or visualized on the co-ordinate plane. For example, consider the equation $3x+y=3$ as

[3] For an illustration of the variety of statements which can be graphed on the number line, see *ibid.*, p. 201.

a statement about ordered pairs (x, y). The ordered pair $(4, -9)$ is associated with a point of the graph of the given equation since "$3(4)+(-9)=3$" is a true statement. We say that the equation is satisfied when (x, y) is taken as $(4, -9)$ or, in other words, when $x=4$ and $y=-9$. There are also many other ordered pairs such as $(2, -3)$ and $(-1, 0)$ that satisfy the given equation. In general, we *solve* a statement about an ordered pair of real numbers x and y by determining a set of ordered pairs of real numbers such that

1. Every member of the set satisfies the statement, and
2. Every ordered pair of real numbers which is not in the set does not satisfy the statement.

The *graph of a statement* is the set of points associated with the set of ordered pairs which satisfy the statement. The set of all ordered pairs which satisfy the statement is *the solution of the statement*. As in the case of statements involving a single real number x the solution is unique but may contain many elements.

A verbal description of the solution of a statement about ordered pairs of real numbers (x, y) often requires a well-developed concept of functions and therefore may not satisfy or be appreciated by students in the early grades of the secondary school. For example, we may properly describe the solution of $3x+y=3$ as the set of ordered pairs $(p, 3-3p)$, where p is any real number. This statement of the solution satisfies the above two conditions but it may not be easily interpreted by some students. Accordingly, it is often desirable to use a geometric interpretation or description. For example, in the case of the equation $3x+y=3$ we might simply plot the graphs of a few of the ordered pairs that satisfy the equation and then take the customary intuitive step of asserting that the graph of the equation is the straight line which passes through the plotted points.

Up to this point in our discussion of graphing on the Cartesian plane, we have essentially followed the customary presentation of this topic. The locus concept has been developed and the students are now ready for practice. After a reasonable amount of practice in graphing equations, most students will begin to develop short cuts or mechanical procedures for plotting graphs, e.g., finding intercepts or locating points with integral co-ordinates. In some cases the teacher introduces the slope-intercept method. We have no quarrel with short cuts. Indeed, students should be encouraged to discover them and to understand the mathematical reasons for them. We have no quarrel with the teaching of the slope-intercept method, for this topic is an exceedingly important one in elementary mathematics. However, we do contend that a student has little opportunity to examine and re-examine the locus concept when he applies his short cuts. Accordingly we now consider several types of exercises which may be treated along with the usual linear equations. A wide variety of exercises is recommended even in elementary algebra, since, when this is done, the students are required to pay careful attention not only to the symbols and relations used, but also to the locus concept. In other words, fundamental concepts of both algebra and geometry may be emphasized in this way.

The first few examples have been considered in detail in order to emphasize our approach to the graphing of statements about pairs of real numbers.

Example 1. Graph $y>x$. (See Fig. 3.)

The reasoning used by the student would be expected to include essentially the following steps:

1. Consider a few points on the graph, e.g., (2, 4), (3, 4), (0, 4).
2. Make one or more false conjectures such as, "All points on the line $y=4$ satisfy the given inequality."
3. Recognize that the graph is not a straight line, but many of the points

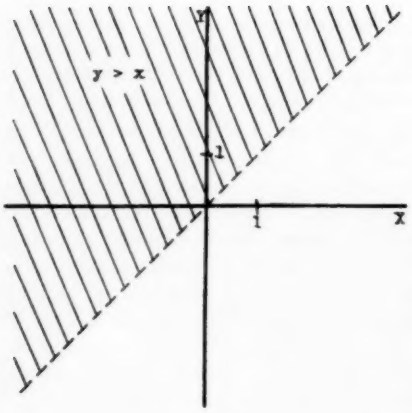


Figure 3

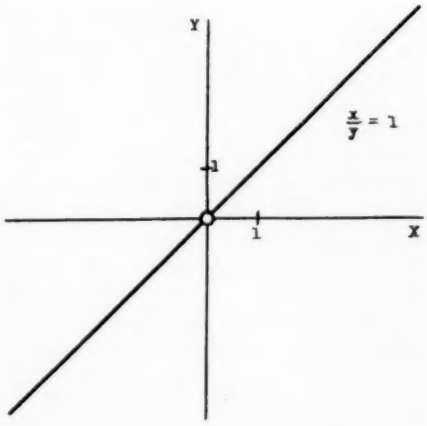


Figure 4

on a line may satisfy the inequality. For example, all the points on the left of (4, 4) and on the line $y=4$ are points on the graph.

4. Recognize the special role of such points as (4, 4), (3, 3), (0, 0); draw the line $y=x$, observe that the points on this line are not points of the graph of $y>x$.
5. Conjecture that all points above the line $y=x$ are the graph of $y>x$. Test this conjecture for several points.
6. Test the conjecture for arbitrary points that are above the line $y=x$ and on such lines as $x=1$ and $x=5$, thereby convincing himself that his conjecture is true for points in the first quadrant.
7. Realize that $y>x$ for all points in the second quadrant.
8. Test the hypothesis for points in the third quadrant.
9. Feel confident that all points above the line $y=x$ are on the graph of $y>x$.
10. Convince himself that no point on or below the line $y=x$ is a point on the graph of $y>x$.

Example 2. Graph $x+y=y+x$.

In view of the experience he has had in graphing linear equations, the student would be expected to consider an algebraic simplification of the given statement. His thinking might proceed as follows:

"I'll collect like terms. . . . What's this? $0=0$. The origin? I'll check. Sure, that works. Wait a minute, let's go back to the original equation. What does it say? Find points for which the x-co-ordinate plus the y-co-ordinate equals the y-co-ordinate plus the x-co-ordinate. That's silly because that's always the case. Ah-hah, *every* point works. I've got it. The graph is the whole plane. Shade the whole thing. Really need a paint brush to draw this graph."

Example 3. Graph $x/y=1$. (See Fig. 4.)

The student's thinking might develop as follows as he solves this problem:

"I'll write '$\times y$' on each side. That gives me $x=y$. I've had this equation before. Its graph is a line through the origin, upper right to lower left. Does that make sense? I know that these points have equal co-ordinates. Of course, when the numerator and denominator are the same, the fraction stands for 1. $3/3$ is 1; $-6/-6=1$. When the terms are different, the fraction doesn't give 1. This looks too easy. Where's the trick? Oh yes, can't have 0 as denominator. That means (0, 0) doesn't work. I'll just put a little circle around the origin to exclude it. But (0, 0) satisfies $x=y$. Something's funny here. What did I

do wrong? I rewrote the given equation $x/y=1$ as $x=y$. The point (0, 0) is on the graph of $x=y$, but is not on the graph of $x/y=1$. Of course, now I see it. When I transformed the original equation, I wrote '$\times y$' on each side, and I was really assuming that y could not be 0 because the two equations are not equivalent unless the transformation is reversible. Yes, and x can't be 0 either, because 0 divided by any number, except 0, is 0 and not 1. Hmm, I wonder what the graph of $x/y=0$ looks like. . . ."

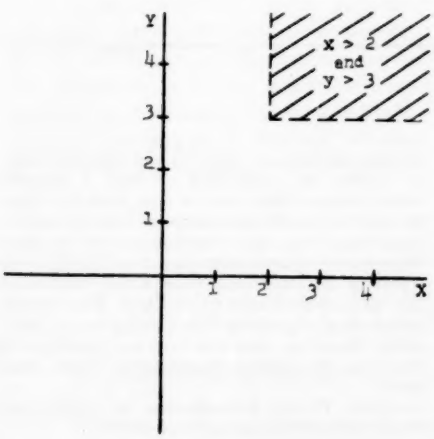
The graph of $x/y=1$ is the set of points different from (0, 0) and on the line $x=y$.

Example 4. Graph the statement "$x>2$ and $y>3$." (See Fig. 5.)

Students usually recognize that such a statement contains two conditions and reason somewhat as follows:

"What does this tell me? Every point on the graph must have an x-co-ordinate greater than 2 and a y-co-ordinate greater than 3. Let me take a look at the first part, $x>2$. That condition is satisfied by every point on the right of the point . . . no, on the right of the line $x=2$. This whole region here. Now, $y>3$ means every point above $y=3$. But I'm looking for the points that do both. I see. The graph of the statement "$x>2$ and $y>3$" is the set of points that are on the right of the line $x=2$ and also above the line $y=3$. Neither the points on the lines themselves nor the points on the other sides of the lines satisfy the conditions. The problem is solved."

Figure 5



Example 5. Graph $|x+y|<0$.

In this case the student may experience a momentary confusion:

"Hmm, absolute value. Absolute value of some number is less than 0? That can't be. No matter what point I pick, the sum of its co-ordinates must be positive or negative or zero. And the absolute value will be either positive or zero. So there are no points whose co-ordinates satisfy the given statement. That means there are no points on the graph. What did he call that? Ah, I remember; I'll just write 'empty set' on my paper."

The graph of $|x+y|<0$ does not contain any points. There are no pairs of real numbers satisfying this condition.

6. $xy=0$
7. $x+y<2$
8. $x=1$ and $y<1$
9. $x^2=4$
10. $x^2<4$
11. $y^2>0$
12. $x-y=y-x$
13. $x-y=x-y$
14. $|x-y|=x-y$
15. $|x-y|=1$
16. $xy>0$
17. $xy<0$
18. $|xy|<0$
19. $|xy|>0$
20. $xy=yx$
21. $|xy|=yx$
22. $|xy|>yx$
23. $|xy|<yx$
24. $\frac{x}{y}>1$
25. $\frac{x}{y}=\frac{y}{x}$
26. $\frac{|x|}{|y|}=\frac{x}{y}$
27. $\frac{|x|}{|y|}\neq\frac{x}{y}$
28. $|\frac{x}{y}|=\frac{|x|}{|y|}$
29. $|\frac{x}{y}|\neq\frac{|x|}{|y|}$
30. $|x+y|=|x|+|y|$
31. $|x+y|<|x|+|y|$
32. $|x+y|>|x|+|y|$
33. $|x-y|\geqq|x|-|y|$

34. $\frac{x}{y-1}=1$ 35. $\frac{x-1}{y}=0$

36. $\frac{1}{x}=\frac{1}{x}$ 37. $\frac{1}{|xy|}=\frac{-1}{xy}$

38. $x<7$ or $y>3$ 39. $x<7$ and/or $y>3$

40. $x\neq 5$ and $y\neq -2$ 41. $x(x-2)>0$

42. $x(y-1)=0$ 43. $x(y-1)>0$

Further applications

When students have developed the ability to draw the graphs for statements such as those given above, they have the background necessary for further interesting and challenging problems and applications. We have space here to suggest but a few of these.

The *graph of a system* of equations may be defined as the set of points common to the graphs of the individual equations. The various types of systems of equations (consistent, inconsistent, indeterminate) can be very advantageously studied from the point of view of intersections of sets of points.

The student may also be asked to find the area of that portion of the Cartesian plane which coincides with the graph of a system of inequalities such as the statement "$x>0$, $y>0$, and $x+3y<6$." Or a student may be given a closed geometric figure and asked to write a system of equations and/or inequalities the graph of which is the region bounded by the figure. The teacher can readily devise problems which are very easy as well as problems which will challenge the best students.

Summary

In this paper we have endeavored to show how graphing may be used to facilitate the student's understanding of algebraic concepts related to pairs of real numbers. We emphasized the locus concept of plane geometry as an important aspect of a study of statements about real numbers in elementary algebra. We could also have emphasized the function concept, but this was not our present aim. Instead we made use of visual and intuitive geometric concepts as we considered the solution of several types of statements. The students were encouraged to base their reasoning on mathematical principles—both algebraic and geometric. Teachers were encouraged to discuss a wide variety of problems so that the students would need to rely upon basic concepts. Our emphasis upon both algebraic and geometric concepts stems from our firm conviction that an understanding of mathematics rather than a narrow view of any one of its branches should be the goal throughout all mathematics courses.

"In some textbooks of elementary mathematics, particularly the less recent ones, one does occasionally come across formulations which convey the impression that it is possible to attribute an independent meaning to variables. Thus it is said that the symbols 'x,' 'y,' . . . also denote certain numbers or quantities, not 'constant numbers,' however (which are denoted by constants like '0,' '1,' . . .), but the so-called 'variable numbers' or rather 'variable quantities.' Statements of this kind have their source in a gross misunderstanding. The 'variable number' x could not possibly have any specified property, for instance, it could be neither positive nor negative nor equal to zero; or rather, the properties of such a number would change from case to case; that is to say, the number would sometimes be positive, sometimes negative, and sometimes equal to zero. But entities of such a kind we do not find in our world at all; their existence would contradict the fundamental laws of thought. The classification of the symbols into constants and variables, therefore, does not have any analogue in the form of a similar classification of the numbers."

—*Alfred Tarski*, Introduction to Logic (*New York: Oxford University Press, 1950*).

Some comments on a simple puzzle

GERALD R. RISING, *Brighton High School, Rochester, New York.*

Puzzle situations stimulate interest
if not used too frequently.
This puzzle, while not new, may remind some teachers that
mathematics can be taught through its use.

MANY TIMES teachers find it difficult to relate textbook situations to situations beyond the covers of the text. A great deal of effort has gone into devising problems for courses at all levels in order to make them more closely related to real life situations. Often this effort falls short, and the student continues to treat text and classroom as worlds apart. Many times, too, situations created through great effort fail to dent the armor of today's students, because of what seems to be their increasing sophistication. Few have failed to witness, for example, the marked change in student reaction to motion pictures in the classroom.

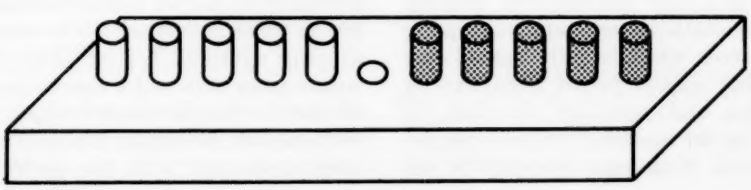
The other day a girl in one of my sophomore high-school mathematics classes brought in a simple puzzle which, she announced, a friend of her father had "solved mathematically in ten minutes." Although the mathematical demonstration to which the girl alluded was beyond the ability level of the students I was teaching at the time, I could hardly fail to note the quickened interest of class members as we considered for a minute a situation usually outside the realm of cold mathematics.

This article is a discussion of the problem posed by the simple puzzle referred to above. The problem itself is not unique or even special. But when *properly used*, this puzzle, and perhaps others similar to it, can create situations not often achieved in school. This particular puzzle-problem is related to topics usually found in college algebra or advanced high-school algebra. Other puzzles may well be sought out for lower levels.

The puzzle is available as a merchandized product. It consists of a plastic stand in which eleven holes are drilled. Ten pegs that fit the holes, five red and five blue, accompany the plastic stand. An examination of the rules of the game show that coins and a sheet of paper on which eleven circles have been drawn may easily be substituted.

The game is started with pegs of the same color grouped at each end with the hole in the center left empty as shown in Figure 1. The directions stated on the manufactured puzzle are as follows:

Figure 1



Object of game is to reverse the positions of the two sets of pegs. That is, if red pegs start at left, they should finish at right. Blue pegs, vice versa.

Start with center hole open. Pegs of both colors may either move one space or jump one peg. Jumps of more than one peg NOT permitted.

Pegs of both colors can only move *forward*, never *backward*.

Only a few minutes are required to solve the problem as given. If there is a single key, it might be to keep colors alternate in the middle of the board. Actually little skill is needed to move correctly.

An equation may be evolved to show the relation between the number of pieces of each type and the number of moves necessary to complete the problem. This may be done for one through five pieces of each using the board in Figure 1 if end pegs are removed. The following table of values is discovered by tabulation:

Number of pegs of one color (x)	1	2	3	4	5
Number of moves to complete problem (y)	3	8	15	24	35
Δy		5	7	9	11
$\Delta^2 y$			2	2	2

The first and second differences included here indicate a quadratic relationship. Substituting in the general type quadratic ($y=ax^2+bx+c$) gives for the first three value pairs: $3=a+b+c$, $8=4a+2b+c$, and $15=9a+3b+c$. Solution of this system of equations and substitution in the general quadratic produces the equation for our puzzle: $y=x^2+2x$, or in factored form, $y=x(x+2)$. (Note that this equation is simpler than the $y=2^x-1$ of the game usually called Tower of Hanoi.)

*

Here is a neat circumscribed problem requiring a little dexterity at manipulating puzzle pieces, which is easily learned during a brief practice period. Some care in tabulation and standard procedure in setting up an equation complete the requirements. With care exercised in describing the puzzle and its rules, it could be proposed as a *textbook* problem. As mentioned previously, circles on a piece of paper and coins of two types could be described instead of the manufactured puzzle, if such puzzles are not available to the students.

Depending on the ability level of students asked to solve the mathematical wrinkle involved here, some of the following questions could be proposed: (1) Determine the number of moves necessary to solve this puzzle with one to five pieces of each color; (2) Make a table of values showing the relation between the number of pieces of one color and the number of moves required to complete the problem; (3) By means of successive differences, determine the type of equation relating these variables; and (4) Derive an equation showing the relationship.

It must be noted that solution time would probably be divided about equally between "non-mathematical" busy work and mathematical technique application. In the latter steps the student begins to do what most texts ask him to do in the first place. (How often do we see in our text such statements as, "From this table of values. . . . "?) This puzzle starts the student at the true beginning of a problem in an interesting situation similar perhaps to situations he might meet outside the classroom. These two factors, the interest of the student and the proximity to possible real situations, overshadow in my opinion the fact that steps in working out the problem are "wasted effort."

The fact that such a puzzle provides only a pseudo-realistic situation does not detract from its value as a mathematics problem. Witness in this regard the interest in many types of puzzles shown by a number of famous mathematicians, interests which sometimes led to important scientific advances. It is an excellent substitute or supplement for the mathematics laboratory. Usually when we think of the mathematics laboratory we are immediately confronted with the problems of

preparation time, equipment to work with, and supervision. The puzzle is a shortcut.

Perhaps coming across such a problem as this in an assignment would be too much of a change for the student, and a good deal of the value would be lost. There is no question that it would be more satisfactory if students, unasked, brought such puzzles to class at the proper moment. Until such a happy time comes, the teacher will have to introduce the problem himself.

Have you read?

BEHARI, RAM. "Presidential Address," *The Mathematics Student*, January–April 1955, pp. 39–57.

If you are interested in the problems of mathematics on the other side of the world, you will want to read this presidential address. It was given before the Indian Mathematical Society in December, 1954. It is interesting to note the parallelism between India and the United States. New mathematics demands new approaches. You will appreciate the statement that "every major concern among intellectual concerns of man is a concern of mathematics." Do these problems in India sound familiar: (1) outmoded syllabi; (2) scarcity of inspiring teachers; (3) lack of research facilities; and (4) low pay?

You may want to try the suggested remedies: (1) A teacher's duty is to discover and nurture mathematics talent; (2) Scholarships must be found for the worthy; (3) Centers of work must be established so that ideas can be exchanged; (4) More incentive must be given for creative work.

You may be surprised to learn that the President feels the two major problems are that of (1) mathematics instruction in the light of the needs of social scientists, and (2) the clarification of the language of advanced mathematics.

As is usually the case, this presidential address gives a great deal of good historical material, especially that dealing with geometry. You and your students will enjoy it.

EZELL, L. B., and TATE, HENRY HAROLD. "High School Students Look to the Future," *Journal of Educational Research*, November 1955, pp. 217–222.

This is the report of a research carried on in several types of schools involving 1,572 boys and girls. As teachers, we need to know all we can about where our boys and girls are going after completion of high school. This study tries to answer this question. A few of the findings are:

3/5 of Boys and 1/2 of Girls plan to attend college.

1/4 of Boys and 1/5 of Girls plan to go directly to a job.

1/10 of Boys and 1/4 of Girls plan to enter trade schools.

None of Boys and 1/50 of Girls plan to marry immediately after high-school graduation.

What vocations do they hope to enter?

43 per cent of Boys and 42 per cent of Girls plan to enter a profession.

13 per cent of Boys and 10 per cent of Girls plan to enter a trade.

3 per cent of Boys and 32 per cent of Girls plan to enter clerical work.

There is a high relationship between wishes of boys and girls and the education of their parents. The desire to go to college seems to vary with the grade in high school; the largest percentage is found in the Sophomore class.

This is interesting information, and after you read it, you may want to make a similar study in your school.

TUKEY, J. W. "Mathematical Consultants, Computational Mathematics and Mathematical Engineering," *American Mathematical Monthly*, October 1955, pp. 565–571.

Although this article is directed toward the programs of doctoral candidates in university mathematics departments, it contains valuable information for the high-school teacher. It helps answer the question as to what vocations the study of mathematics will admit them. What effect will high-speed calculators, such as I.B.M. 701, have on the future professional mathematician? Is there a solution to the perennial problem of engineering mathematics *vs.* theoretical mathematics?

The author tries to answer these questions by consideration of the characteristics and training of one who would enter the profession. Some of the areas discussed are mathematical consultants, computational mathematics, theoretical mechanics, mathematical engineering, and computing engineering. Suggestions for mathematics departments will interest you and be of value. —PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

The place of evaluation in the secondary school program

MARIAN C. CLIFFE, *Los Angeles City Schools, Los Angeles, California.*

Evaluating the outcomes of school life is not simple. It involves more than giving an "A" grade in trigonometry.

ONE WINTER night seven or eight years ago snow fell in Los Angeles. Schools were closed the next day, not entirely because it was difficult for pupils to get to school, but because snow was an educational experience, as thousands of boys and girls were discovering the new thrill of walking and playing in snow.

On the second day schools were in session as usual. On the valley floor the snow remained in patches, but in the foothill areas snow still lay three or four inches deep. A principal on his way to a foothill school stopped his car to pick up a junior high school boy who carried a paper sugar bag full of what appeared to be a very large lunch. Driving along looking out across the white expanse, the principal said, "This is really pretty. Have you ever been in snow before?"

"No," replied the boy, "this is the first time I've been in snow in my whole life."

"How do you like it?" asked the principal.

"Gee, I love it!" said the boy. "See, I've got a whole bag of it here that I'm taking to school."

While eastern boys and girls may like snow and enjoy it thoroughly, I doubt if many of them treasure it as did the boy who had never seen it before.

And that brings to mind a factor which enters into any discussion of evaluation. People are engaged in informal evaluation of one sort or another most of the time. To each of these informal evaluations, the evaluator brings much of himself. His past experiences, his set of standards, the mores of his particular social group, even his own achievements color his thinking. And so the little boy's experience of living in a warm climate enhanced his appreciation of snow and made him evaluate it as a treasure. We *do* see through the eyes of our own experience. We *do* evaluate subjectively.

Therefore, it is important that as educators we concern ourselves with the use of objective means of evaluation whenever possible, that we seek to improve existing evaluative instruments, and that we be alert in our search for new approaches to the problem of evaluation.

Actually, objective evaluation is not new. From the beginning of time, man has been interested in evaluative procedures more accurate than mere opinion. Evidence of this is found in tribal rites and customs. As the centuries rolled on, evaluation took other forms and meanings. In the program of the modern school, it occupies an important place.

To the modern educator, what is meant by evaluation? It has been well described as the process of gathering and weighing evidence which will reveal behavior changes of pupils. It is concerned not only with growth in the basic skills but also with growth in the attitudes, appreciations, and knowledge needed for effective living.

In its earliest form, evaluation in the schools involved testing. Now, probably as a result of the child-study approach,

it may include objective tests, measurement, anecdotal records, interviews, questionnaires, rating scales, case study, and class records. We are studying the whole child, we say.

Why do we evaluate? The most commonly emphasized reason, although by no means the most valid, is evaluation for the purpose of assigning "marks" or "grades." But grading as an end in itself is not an acceptable reason for evaluation. We evaluate for the purposes of promotion, ability grouping, reporting to parents.

But the most valid, the most important, use of evaluation in mathematics is as an aid to teaching. By evaluating we discover growth in skills, abilities and understandings. We learn the nature of difficulties which may be present. We are then able to teach more effectively.

In providing a sound educational program for high school pupils, *what do we or what should we evaluate?* First, we are interested in the academic progress of the pupil. We test his computational skill, his accuracy, sometimes his speed. We attempt to test his understanding of processes. We are interested in measuring his problem-solving ability. But usually we are content to *present* the problems to him for solution. Today the ever increasing use of computing machines, the automation of many processes in industry, and the increasing complexity of scientific investigation place a new responsibility upon the teacher of mathematics.

Early in 1955 a personnel director from one of the nation's large aircraft companies spoke to mathematics teachers about the need for engineers. He pointed out his lack of interest in whether or not the men who came to his company could solve complicated sets of simultaneous equations. He said, "Our equations come with 20 or so unknowns. It takes too long to solve them. We feed them into a computer which can solve them in a very short time." He spoke of industry's need for men who can set up the problems which the machines will solve and, more important still, for men who realize that the problems exist.

Today's schools are preparing young people for such jobs, training also the thousands who, because of the ever increasing complexity of our world, will need to think in mathematical terms. The evaluation which is part of this preparation must include evaluation of problem-defining and problem-solving abilities. It must place emphasis upon thinking, upon meanings and understandings.

A second answer to the question as to what we evaluate might make use of that overworked phrase, "the whole child." The truth is that there is more to be considered about John Jones than his academic achievement alone. Frequently his difficulties in his mathematics class stem from problems of physical, mental, or emotional health.

I should like to refer again to a panel of aircraft company experts. Having stressed the necessity for rigorous training, for alert, well-prepared personnel, they indicated the need for one additional attribute; namely, leadership. They concluded that *of the two* they would rather employ a graduate with a "B" average and a record of participation in campus activities than an "A" student who had not participated. They expressed a preference for graduates who were able to earn grades "without running at full steam pressure, without having their noses buried in a book."

The mathematics teacher cannot afford to ignore the implications of this statement, which is by no means unique or unusual. He will recognize his share of responsibility for the social and emotional adjustment of his pupils. Insofar as is possible he will seek to provide a classroom climate in which all pupils feel themselves to be part of the team. He can set the stage so that the intellectually gifted are aware of their share of responsibility for the welfare of the group and every member of the class feels accepted,

free to participate, important in some special way. This is no new thing. It is taking place in thousands of classrooms where good teachers everywhere consider it part of the normal procedure and do not let it detract from the main business at hand; namely, the teaching of mathematics.

Evaluation in the secondary school should not concern itself exclusively with the pupil. The educational program of the school requires evaluation if it is to grow in effectiveness. Are the curricular offerings adequate? Is the course of study in each subject realistic, functional? Is enrichment provided for the gifted? Is provision made for the slow learner or is he lost by the wayside? Are textbooks appropriate, methods the best?

The alert teacher finds also that self-evaluation can be a means of increasing his efficiency and his satisfaction in his job. He finds it profitable to ask himself, once in a while, "Am I giving evidence of professional growth?" "What have I learned during the past year which has a direct bearing on my teaching?" "Am I active in a professional organization?" "What sort of a person am I in the eyes of my pupils?" "Is my voice pleasing, my appearance neat, my manner friendly and interested?" "Do I deal with classroom problems objectively?" "Am I assuming my share of responsibility in the area of moral and spiritual values, the development in boys and girls of desirable character traits?"

What are some of the means used for evaluating the academic progress of the pupil, his total personality, the educational program, ourselves as teachers?

We have already indicated that self-appraisal is one means of evaluation. This method can best be applied by the teacher to himself or by a school system to itself. For example, one large city school system undertook to evaluate its health facilities by means of a questionnaire. The value of this procedure was not in the answers turned in to the central office, but in the improvement which resulted from focusing attention upon existing conditions.

Self-evaluation can be carried too far, particularly when it is pupil self-appraisal of academic or other achievement. We cannot and we do not ask Johnny what he would like to do today and later ask him how well he thinks he did it.

In the specific field of mathematics, evaluation of achievement, growth, competence, and understanding lend themselves rather well to objective testing. Therefore, it is of primary importance that we exercise care and intelligence in the selection and use of testing instruments.

In practice we note the use of teacher-made tests, departmental tests, standardized tests, city-wide tests, and a few "make-it-up-as-you-throw-it-on-the-board tests. The good teacher will use several types of evaluative devices. To various types of testing procedures he will add observation, teacher judgment of pupil growth, conferences, and appraisal of daily work.

For mathematics, there is nothing new in the use of the objective type test. The oldest type is the completion: $7 \times 8 = ?$ To this have been added true-false matching, multiple choice, arrangement, and others.

A new trend is in the increasing use of the standardized test. Dr. Lee Irvin in a study of methods of evaluation found two pronounced and divergent attitudes toward standardized tests. One was a calm acceptance, the other a strong opposition.

She mentions too the interesting fact that teachers in Iowa and other midwestern states list the use of the California Arithmetic Achievement Test, while California teachers make use of the Iowa Every Pupil Test. Apparently confidence in a testing instrument is directly proportional to its distance from home ($C = Kd$).

Brief mention should be made of one other type of evaluative procedure. One large school system sent a questionnaire to a sampling of its recent graduates.

Responses included some which may concern mathematics teachers. If the graduates could repeat their high school careers, their major field of study would in many cases be changed. One department would have a decrease of 50 per cent in enrollment, one would double the number of boys enrolled. Mathematics and science would hold approximately their original numbers. A study of college enrollment showed that twice as many boys as girls went on to college.

For graduates who went to work there was no significant difference in the matter of job satisfaction between those with high I.Q.'s and those with low I.Q.'s. Both groups were equally well satisfied with their jobs. Also, both groups were equally able to find and keep jobs.

In free-responses statements, one-sixth of the graduates asked for improvements in school personnel. Most frequently mentioned was the request for "teachers who take a more personal interest in pupils." One boy (I.Q. 107) stated, "No one was personally interested." Another (I.Q. 83) wrote, "Too many teachers make the students feel it is a real problem to like them." One girl (I.Q. 91) wrote, "Teachers might try to explain a little bit more on how very different life is when you're out of school and how very grown up it is."

On the other side of the ledger were the many who expressed satisfaction and appreciation. "My friendships with teachers," said a boy (I.Q. 122), "were more important than my friendships with pupils." A girl (I.Q. 118) wrote, "The most valuable thing I learned in high school was the imprint of a few teachers upon my life." A boy (I.Q. 86) stated, "I learned two things in high school, good sportsmanship and how to appreciate good music. But it wasn't the courses that taught me this, it was the teachers."

We recognize that we should do all that we can to improve the program of the secondary school. We should make use of evaluative instruments and techniques. Evaluation is a continuous process. Its place in the secondary school program is a most important one. We need to appraise the total program, improve our efficiency, find more ways of meeting pupil needs.

But we should remember that one of the most fundamental of pupil needs is the need to be liked, to have someone really interested in him, someone who can accept him as a person even when his algebra is unacceptable.

For this reason we are reminded that above and beyond all that evaluation, methodology, textbooks, and curriculum can accomplish stands the importance of the individual classroom teacher who daily lives and works with his pupils. Mathematics teachers in general are thoughtful people, intelligent people. Because of that they frequently feel very keenly the discouragement which comes from not being able to accomplish all that their active minds can plan.

It is, therefore, encouraging to the teacher to know that although he may have been so busy he has not done as much professional reading as he wanted to do, nor written the mathematics textbook he dreamed to write, nor engaged in all the professional activities in which he has been interested, yet in the final evaluation he may have made the most enduring contribution of all. He has made this contribution, if in addition to imparting knowledge and abilities, he has given friendship, understanding and inspiration to the boys and girls who daily troop through his classroom door.

The role of sensory materials in meaningful learning

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Learning aids are used frequently, and many aids are available. However, there exists a real need to evaluate the numerous aids available in the light of what can be accomplished through their use.

IF I SHOULD choose a text appropriate to the subject of this discussion, it would be from Ecclesiastes, chapter 1, verses 9 and 10: " . . . and there is nothing new under the sun. Is there a thing of which it is said, 'See, this is new'? It has been already, in the ages before us."

With this text in mind, I would ask if we can lay claim to today's popular use of sensory learning materials? Almost four hundred years ago, H. Billingsley of London prepared an English translation of Euclid's *Elements*, in which appeared paper figures pasted so that when opened they would make models of space figures. This innovation in textbooks precipitated a general trend at that time. Many textbooks appeared with various versions of the so-called "stand-up" models. As early as 1752, John Lodge Cowley of London, an ardent exponent of using devices in the teaching of geometry, wrote several books containing various "movable schemes." Colored lines instead of letters on diagrams were used in a textbook in 1847. This book was probably the first to suggest the use of colored chalk and colored pencils that are currently employed in complicated geometric drawings. Obviously, those authors reflected a need felt for sensory learning materials at that time. Indeed, the text referred to at the outset of this paper bears out the fact that sensory materials have " . . . been already, in the ages before us."

Bearing out the old adage that history eventually repeats itself, the use of sensory materials is found to be a highly favored and popular educational technique today. Evidence of such popularity is indicated by numerous articles written on this topic, the large exhibits at professional meetings, and the very fact that one yearbook of The National Council of Teachers of Mathematics was devoted entirely to the subject of multi-sensory materials. However, upon investigation, it is often found that many of the previously accepted ideas and practices have ceased to have a favored role in our present educational program. The reasons for such changes are varied. One rather common reason which might influence the tenure of sensory materials in the teaching of mathematics is the degree of wisdom exercised in their selection and use.

Assuming there is value to be gained in the use of these materials, an attempt is made to establish criteria that can be used as a guide in their selection. It is assumed that every mathematics teacher wishes to develop the pupil's interest in mathe-

matics, to stimulate his imagination, to assist him in developing an understanding of concepts, relationships, and processes, to encourage him in developing a spirit of investigation and discovery, which will lead to abstract thinking and the formulation of generalizations, and to help him recognize the applications of mathematical principles in practical situations. Many claims are found in current professional literature that the use of sensory materials will aid in achieving these generally accepted goals. Consequently, these goals appear to be a justifiable basis for criteria to be used in selecting sensory materials.

To illustrate the evaluation of a sensory device by using these criteria, the area board, a model commonly used by mathematics teachers to introduce products and factoring, will be used. The area board is composed of a square and three rectangles which can be arranged to form a larger rectangle as in Figures 1-A and 1-B. These figures may be cut out of ply-board and arranged in a frame as any ordinary cut-out puzzle. They also may be cut from paper and backed with sandpaper, so they will adhere to a flannel board. One set of these figures is ruled into square units, while in a similar set the rulings are omitted. This set may be cut from brightly colored flannel, each piece being a different color.

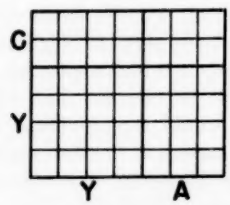


Figure 1-A

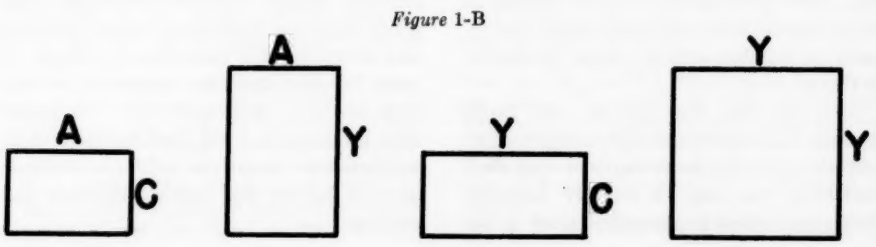


Figure 1-B

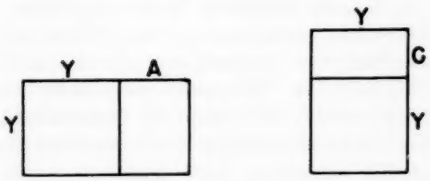


Figure 2

Products are introduced by applying the principle used in finding the area of these rectangular figures, while factors are introduced by using the reverse process of finding their dimensions. The pupil finds this process an easy one when the model contains the rulings. From this he is gradually led to consider the areas of figures where the rulings have been omitted, as in Figure 1-B.

The next step is to consider the area formed when the square and one of the rectangles are arranged as indicated in Figure 2. Observing that the area of the new figure is determined by adding the area of the square to that of the rectangle, the pupil is led to see that $Y(Y+C)$ is equal to this sum. By assembling the remaining parts of the model (Figure 1-A) the pupil readily discovers that the product of $(Y+C)$ and $(Y+A)$ is equal to the sum of the areas of all these parts. Hence, by using various combinations of the model, a pupil is able to discover and understand the product of two monomials, a monomial and a binomial, and two binomials. When he associates factors with dimensions, they will have more meaning for him.

Invariably as a result of this type of development, some pupil will ask: "How do we find the product of $(X-A)$ and $(X+A)$?", or, "How can three factors be represented?" Obviously, his imagination has been stimulated. Pupil-constructed models answering these questions serve as excellent out-of-class activity. Through such activities, the pupil's concepts will be enriched, and he will be building a background for further mathematical information.

To evaluate this device the following questions are proposed: 1. Has it aided the pupil in developing a meaningful concept of the principles involved? 2. Has it stimulated his imagination and interest sufficiently for him to propose other relationships? 3. Has it helped him to recognize the applications of these principles to practical situations? The answers to these questions are left to the reader. An affirmative answer to each should indicate that this particular device can be regarded as one which contributes to the meaningful development of concepts.

Thus, it is possible to select those sensory learning materials that can, when used with discretion, play an important role in meaningful learning. But what caution should be exercised in using sensory materials? It is generally agreed that these materials are very helpful in teaching the slow learner, and the slower the learner the greater is his need for them. One must be aware that the need for sensory materials will undoubtedly vary according to the degree that the student is able to think in abstract terms and to make generalizations without the use of concrete representations—they should not be forced on him. The time used by these pupils in manipulating such materials could well be spent in another activity more profitable to them.

Although the slow learner will profit through his experiences with multi-sensory materials, it must be recognized that their continued use can be equally harmful. Just as a crutch is discarded after it has ceased to be useful, so the aid must be withdrawn as soon as the pupil's development allows. A prolonged use will be weakening.

Another reason for withdrawing the use of the model is that it may cease to teach and become a toy in the hands of the pupil. For example, the elliptic billiard board is excellent to illustrate the focal properties of the ellipse; yet, when it becomes a toy, the learning situation has ceased. Many models are of this type, and the teacher must be alert to the proper time when it is advisable to withdraw them. It has been observed frequently that the interest of a pupil becomes centered on the operation of the device rather than on the mathematical concept or principle.

Only those materials that contribute toward meaningful learning have been discussed. Mention should be made of materials used to entertain, enrich, and create interest. They are found in abundance at National Council convention exhibits and seem to predominate over materials designed for developing meaningful learning. This observation seems to bear out, for the "anti-aids" minded person, that mathematics teaching is being replaced by courses in art, home economics, and shop. The artistic curve stitching, attractive lamp shades, replicas of stained glass windows, wooden dolls with polyhedron heads, all serve to heighten his concern. Indeed, it is highly possible for such activities to become so time consuming that they would replace those learnings normally expected in the mathematics curriculum. However, it is expected that wise teachers, recognizing their role, will insure that they do not hinder class activities. Many teachers feel that they serve best as individual home projects and activities for mathematics clubs. It must be admitted that materials of this type are very impressive, but it is hoped that the point has not been reached where mathematics must be sold to students, as well as to the public, through this medium.

In conclusion, there are two types of sensory materials—those used in developing meaningful learnings, and those used to enrich, entertain, and stimulate interest. Each has a role in the pupil's mathematical experiences; however, each must be used under the guidance of a wise teacher who employs certain safeguards. In fact, the use of any sensory materials should always be regarded as *a supplement to good teaching and not as a substitute for good teaching.* There is no substitute for an enthusiastic teacher who knows how and what to teach.

Let's look at language

Why is it *acre*, and not *square acre?*

Occasionally, pupils do attempt to use the latter expression. They have learned that people measure areas by using unit squares as measuring sticks. And, being familiar with square inches, square feet, square yards, and square rods, pupils acquire just the right set to label *acres* as *square acres.*

But *acre* originally referred merely to a piece of land, seldom, if ever, square in shape. Later, *acre* symbolized a specific unit of areal measure; it was defined, however, as a *rectangle*, not a square.

Let us first examine *acre* as land. In the ancient Sumerian language the word *agar* meant a watered field. Also the Latin *ager* and the Greek *agros* denoted fertile pasture land. Likewise, in numerous Teutonic tongues similar words designated grazeable or arable country. Examples of these include the Gothic *akra*, the Dutch *Akker*, the German *Acker*, and the Anglo-Saxon *aecre.*

The transition from *acre* as a vague word meaning pasturable land to *acre* as a precise word meaning a unit of land, took centuries. Indeed, *God's acre* as a name for a burial ground still exists. And *Long Acre*, the site of St. Martin's church, endures. Also we still use *acres* to connote land, estate, or farm.

However, as men worked their fields with oxen, they found it convenient to refer to the area they could plow in one day as an *acre.* The word shifted from land loosely, to land as an amount.

Such a unit, of course, varied considerably. Pupils who consider the story of the word *acre* can readily adduce reasons why "a day's plowing" could mean many things.

Gradually *acre* came to signify a piece of ground 40 poles long and 4 poles wide. Statutes during the reigns of three English kings (Edward I, Edward III, and Henry VIII) thus defined it. We still use 160 square poles (rods) as the legal acre.

How did poles, or *rods* enter the scene? They originally were wooden sticks that the plowmen used to drive their oxen. As is reflected in ancient fables and religious tales, 40 was a popular number. And 40 rods was a convenient distance to plow before letting the oxen rest a bit.

But actually the plowers did not measure off 40 rods. At least they didn't at first. Instead, they plowed until the oxen tired. That distance, literally a furrow long, later became the standard *furlong.* It was easy eventually to limit the furlong to 40 rods, especially since the distance seemed reasonable and 40 was a much-used number anyway. Apparently the *mile*, which had varied widely since antiquity, was fixed at 8 furlongs, or 320 rods.

The rod also provided a convenient way to mark off the distance at which a first return furrow might be plowed. Moreover, to plow up completely 4 such plots was generally a day's work.

This was all very simple, and 160 square rods per acre stuck. But $\sqrt{160}$ is irrational. So, also, of course, are $\sqrt{4840}$ and $\sqrt{43560}$, wherein the radicands indicate the number of square yards and the number of square feet, respectively, in an acre. Accordingly, the idea of a square containing an acre and having its side expressible as an integral number of well-known linear units perishes.

The history of *acre* also sheds some light on the awkward $16\frac{1}{2}$ feet, or $5\frac{1}{2}$ yards, per rod. Whereas 40 rods per furlong and 8 furlongs per mile fit nicely, the number of yards per mile, 1,760, is not an exact multiple of 320, the number of rods in 1 mile. A similar kind of statement applies to 5,280, the number of feet in 1 mile, and 320. Hence, pupils have to deal with $16\frac{1}{2}$ and $5\frac{1}{2}$ instead of integers, which most learners probably would prefer.

Pupils sometimes find further interest in the fact that, although the foot and the yard originated from measurements of the human body, the rod, the furlong, and the acre stemmed from measurements of work.

So, the notion of a square acre lacks the sanction of custom and usage. The acre arose as a rectangle, not as a square, and our desire to compute areas as squares forces us to use square rods or square miles, either of which can be converted readily into acres.

Possibly during this discussion about the word *acre* the question, "How about *wiseacre*?" has arisen. The answer to that one is brief. *Wiseacre* is not related etymologically to *acre.* Instead, it developed from the German word *Weissager*, a foreteller, prophet, or fortune-teller. It now applies, though, to a pretender to wisdom.

—I. H. Brune, Iowa State Teachers College.

• DEVICES FOR A MATHEMATICS CLASSROOM

Edited by Emil J. Berger, Monroe High School, St. Paul, Minnesota

An electric matching device

Clarence Olander, St. Louis Park High School, St. Louis Park, Minnesota

Mathematics teachers are sometimes faced with the problem of having to dissolve attitudes of indifference toward their subject. One effective way of doing this is through the use of mathematical games. If the game is carefully and cleverly designed, the student usually becomes motivated and realizes that mathematics can be fun.

In this article we suggest a matching game which may function as a bulletin board display. The device is so constructed that a student may, by manipulating two wires, cause a light to flash on if he is able to effect a correct matching of two mathematical facts. The experience of this contributor indicates that the action provided by this game evokes much more interest from students than the usual static-type bulletin board. The device which we have in mind may be built quite simply, or more elaborately, depending on the availability of appropriate materials.

To construct the device select a piece of $\frac{1}{8}''$ masonite 4′ by 2′. Paint the masonite with white enamel. Drill a series of ten $\frac{1}{4}''$ holes $1\frac{1}{2}''$ apart, parallel to and 1″ from a side edge (Fig. 1). Drill a second series of ten holes parallel to and at a distance of $2\frac{1}{2}'$ from the first series. Insert an electrical jack in each of the twenty holes. Paint in black the numbers *1–10* alongside the first series of holes, and in red, the letters

Figure 1

Can YOU Make The RIGHT Connection?
1. OBTUSE ANGLE
2. PARALLEL LINES
3. PERPENDICULAR LINES
4. RADIUS
5. HYPOTENUSE
6. ACUTE ANGLE
7. COMPLEMENTARY ANGLES
8. ADJACENT ANGLES
9. DIAMETER
10. ALTITUDE
MR. BRITE NOSE
a. An angle less than a right angle
b. A line from the center of a circle to a point on the circle
c. Two Angles whose sum is 90°
d. Two angles that have the same vertex and a common side between them
e. A line from a vertex of a triangle perpendicular to the opposite side
f. Two lines that meet to form right angles.
g. A line thru the center of a circle with its ends on the circle
h. Lines that lie in the same plane but do not meet.
i. An angle greater than a right angle and less than 180°
j. The side opposite the right angle in a right triangle

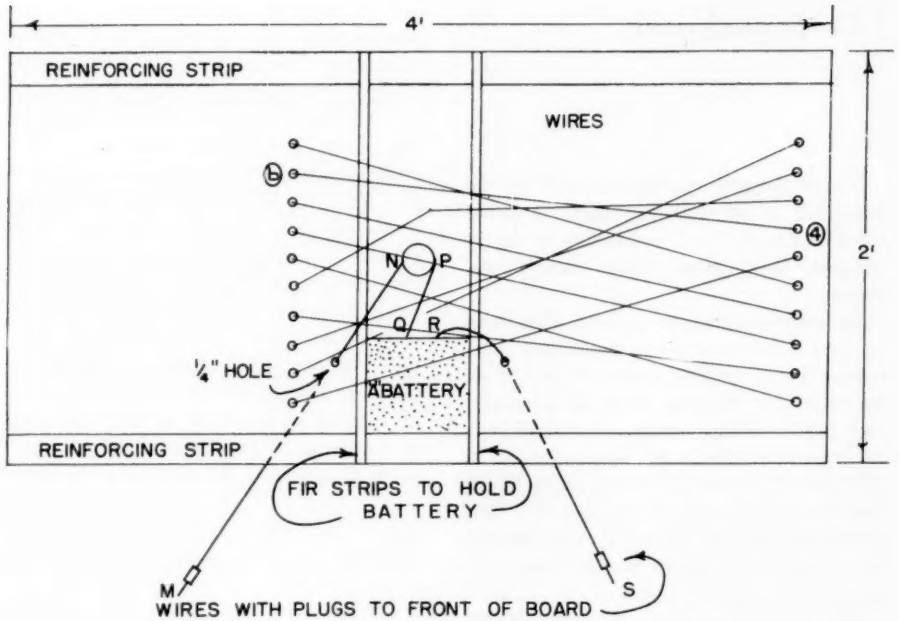


Figure 2

a–j alongside the second series. In the center of the masonite drill a $\frac{3}{4}''$ hole and insert a light bulb with attached terminals and reflector. To make the device more attractive paint in a clever caption and the face of a clown in such a position that the light bulb will serve as the nose (Fig. 1). To fasten the two panels of poster paper on which the matching items are written, drill holes in the masonite near the corners of each panel and use peg board clips as fasteners.

To complete the electrical circuit drill two $\frac{1}{4}''$ holes near the center of the board (perhaps in the tie of the clown). Through each hole insert an insulated wire about 2′ long. Fasten an end of one wire (*MN*) to a terminal of the light bulb and an end of the other wire (*RS*) to a terminal of an "A" battery attached to the back of the masonite (Fig. 2). Connect the other terminal of the battery to the light with a short piece of wire (*PQ*). On the free ends (*M* and *S*) of each of the wires (*MN* and *RS*) fasten electrical plugs which will fit the jacks. Next, cut ten pieces of insulated wire, each about $2\frac{1}{2}'$ in length. To the ends of each, attach small electrical clamps. Use each of the ten wires in turn to connect (on the back of the board) a jack from one series with a jack in the other series (e.g., connect jacks *4* and *b*). The combinations which are used must correspond with correct matchings of the items listed on the panels. In playing the game the student is expected to "make the right connections." If he is able to make a correct matching, the light will flash on.

The device may be reinforced on the back with "reinforcing strips" and fastened to the wall if desired.

The device described is suitable for use at all levels of instruction, since the panels can and should be changed often with different sets of matching exercises. Each time the panels are changed the combinations should also be changed. This is easily accomplished by re-pairing the connections between the two series of jacks.

The tomahawk

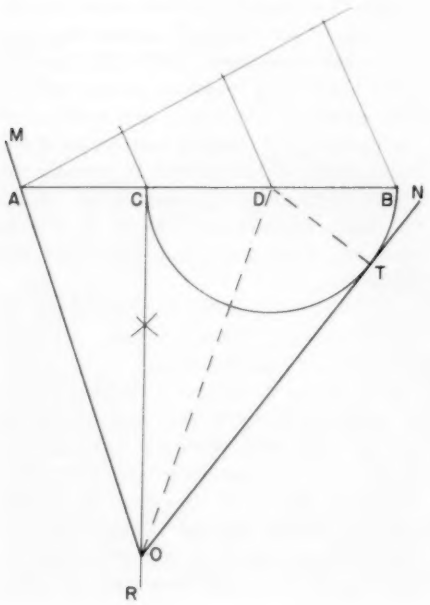
by Bertram S. Sackman, Albany, New York

One of the implements invented for the trisection of an arbitrary angle is called the tomahawk. Its construction is easily effected with straightedge and compasses.

By applying the conventional Euclidean method for trisecting an arbitrary line, locate the trisection points C and D on segment AB (Fig. 3). With D as center and DB as radius, construct a semicircle as illustrated. Draw RC perpendicular to AB at C. The parts of the drawing which form the tomahawk are segment AB, perpendicular RC, and the semicircle. Reproduce these elements of the drawing on a sheet of transparent paper.

To trisect an arbitrary angle NOM place the tomahawk on the angle in such a way that RC passes through vertex O, A lies on side OM, and the semicircle is tangent to side ON at T. If the foregoing conditions are fulfilled simultaneously, then angle $MOC = \frac{1}{3}$ (angle MON). To prove the trisection, draw auxiliary lines OD and DT as indicated.

Figure 3



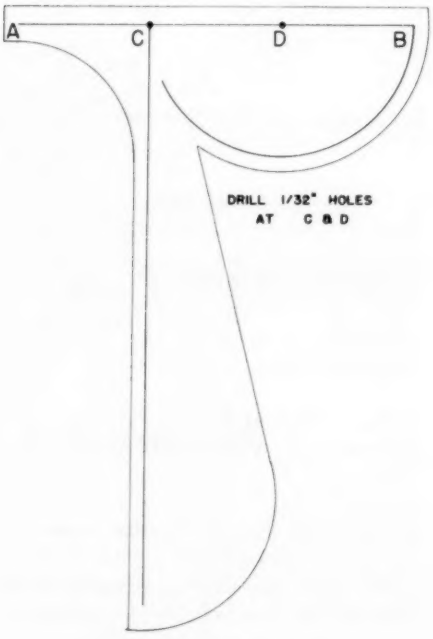


Figure 4

By transferring the construction to a piece of 1/16″ transparent plastic, a template may be developed for trisecting any number of angles. Figure 4 shows what such an instrument might look like. A template whose over-all dimensions are 6″×8″ is rather effective for making multisections.

If a number of congruent tomahawks are used in combination, it is possible to make multisections that cannot be done with straightedge and compasses.

For example, with the aid of three congruent tomahawks the regular heptagon

becomes constructable. By using four congruent tomahawks, the regular nonagon may be constructed, and with eight congruent tomahawks one may construct a regular seventeen-sided polygon.

To show how such constructions can be made, we will describe the procedure for dividing an arbitrary angle XOY into seven equal parts (Fig. 5). Fix point D on the second tomahawk to point D on the first tomahawk. Fix point F on the third tomahawk to point F on the second tomahawk. Now move the tomahawks into such a position that A lies on OX; CR, ES, and HQ all pass through O, and the semicircle with K as center is tangent to OY at T. Mark the point K as shown in Figure 6, and draw KO. Angle KOY is one-seventh of angle XOY.

By this time the reader may have guessed that the principle just described

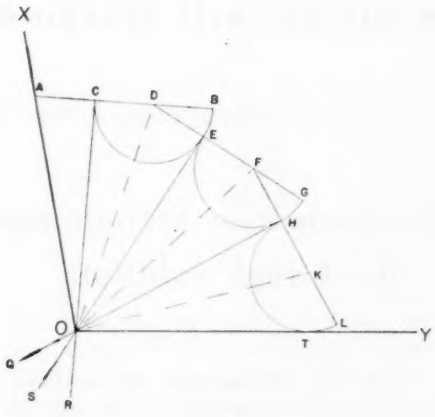


Figure 5

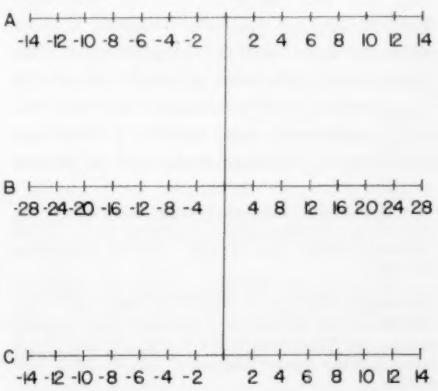
can be used to divide an angle of 90° into seven equal parts. If one of these parts is doubled and then doubled again, the resulting angle is one-seventh of 360°.

How to draw an addition and subtraction nomograph

by Donovan A. Johnson, University of Minnesota High School, Minneapolis, Minnesota

On a sheet of plain paper draw three parallel horizontal lines approximately three inches apart. Label these lines A, B, and C as illustrated in Figure 6. Draw a vertical line at the middle of the sheet. Lines A and C should be numbered in even spaces to the left and right of the vertical line which represents zero. All numbers on the left are negative, and all numbers on the right are positive. The center line, B, is also numbered, but each mark on this line has a numerical value twice that of an equivalent distance on scales A and C. If a straightedge is used to connect any number on A with a number on C, the sum of these numbers may be found by noting the value of the point on B which lines up with the intersection of the straightedge $(A+C=B)$. In a similar manner this nomograph can be used for subtraction $(A=B-C)$.

Figure 6



Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

Irrationals or incommensurables III—the Greek solution

Both the Pythagorean number-theoretic proof of the irrationality of $\sqrt{2}$ (or of the incommensurability of the side and diagonal of a square) and the successive subtraction process which we applied to the square and the pentagon in earlier articles[1] required generalization before one could claim that mathematicians had a theory of irrationals. Some generalizations developed quite rapidly. We shall merely sketch in the steps in this development and explain the finished product as it appeared in Euclid.

The chief contributors to this development were Theodorus of Cyrene (c. 400 B.C.), Theaetetus (c. 375 B.C.), and Eudoxus of Cnidos (c. 400–347 B.C.). Theodorus showed the irrationality of the non-square integers from 3 to 17. Exactly how he did this is not certain, nor why he stopped with 17; but the fact that he treated only these cases indicates that he still did not have a general process.[2] Theaetetus, on the other hand, is supposed to have been the original source for such general theorems as Euclid, Book X, Proposition 8: *If two magnitudes have not to one another the ratio which a number has to a number, the magnitudes will be incommensurable;* and Book X, 9: *The squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number;* . . . ; and *Squares which have not to one another the ratio which a square number has to a square number will not have their sides commensurable in length either.*[3]

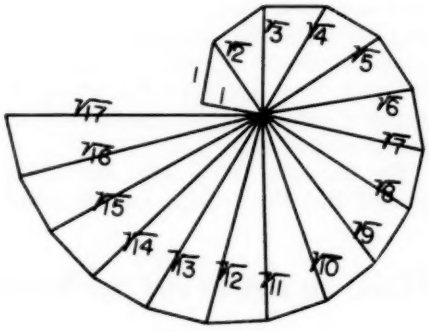


Figure 1

The use in these propositions of the words "magnitude," "line," and "squares on lines" as different from "numbers" indicates that "number" was still "integer" to the Greeks. It still had not occurred to them to extend the number system to include new numbers—irrational numbers—in order that the ratios of all magnitudes could be represented by numbers. However, there is at this time explicit recognition of that which was a "logical scandal"

[1] P. S. Jones, "Irrationals or Incommensurables I, II," The Mathematics Teacher, vol. XLIX (February 1956), pp. 123–127, (March 1956), pp. 187–191.

[2] B. L. van der Waerden, in *Science Awakening* (Groningen, Holland: P. Noordhoff, Ltd., 1954), pp. 145–46, summarizes several interesting theories which explain why Theodorus stopped with $\sqrt{17}$, one reason being that the special diagram of Figure 1 laps itself at 17.

[3] T. L. Heath, *The Thirteen Books of Euclid's Elements* (Cambridge: The University Press, 1908), vol. III, p. 28.

to the Pythagoreans—the existence of magnitudes, areas or lines, which did not have ratios representable by integers.

Eudoxus provided the definition of equal ratios or proportion and the method for using the definition which for centuries and until relatively recent times was the tool for dealing with the ratios of incommensurable quantities without the use of irrational numbers. This definition as stated in Euclid's Book V, Definition 5, was: *Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equal multiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.*[4] This says that $a:b=c:d$ if and only if

$$ka>mb \text{ implies } kc>md$$
$$ka=nb \text{ implies } kc=nd$$
$$ka<pb \text{ implies } kc<pd.$$

An illustration of the use of this definition is to be found in the proof of the theorem: *Triangles and parallelograms which are under the same height are to one another as their bases.* In Figure 2, ABC and ACD are two triangles with the same height. Produce BD in both directions laying off any number of segments $BG=GH=BC$ and $DK=KL=CD$. Complete the triangles with these segments as bases and with vertex at A. Therefore triangle AHC is the same multiple of ABC that base HC is of BC.

Similarly triangle ADL is the same multiple of ACD as DL is of CD.

"Thus there being four magnitudes, two bases BC, CD, and two triangles ABC and ACD, equimultiples have been taken of the base BC and the triangle ABC, namely the base HC and the triangle AHC; and of the base CD and the triangle ADC other, chance, equimultiples, namely, the base LC and the triangle ALC;

[4] *Ibid.*, vol. II, p. 114.

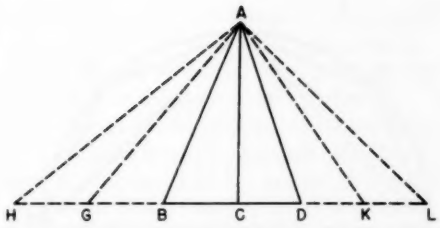


Figure 2

and it has been proved that if the base HC is in excess of the base CL, the triangle AHC is also in excess of the triangle ALC; if equal, equal; and if less, less. Therefore as the base BC is to the base CD, so is the triangle ABC to the triangle ACD."[5]

The proof for parallelograms was similar. Today, as then, logically some recognition of the possible irrationality of the dimensions of geometric figures must be made at the time of defining their areas or the ratios of areas. This may be done in several ways. One procedure is to postulate a correspondence between the "lengths" of lines or between points on lines and the real numbers (which of course include irrationals) and to assume the ability to operate with irrationals. A second procedure is to use a limit idea explicitly in getting the ratio of a figure, e.g., a rectangle, to a figure defined to have a unit area. Of course and unfortunately, there has also been a tendency to ignore the whole question in secondary school instruction.

Both Eudoxus' definition of proportion and his "method of exhaustion" are illustrated by Euclid XII, 2, which says: *Circles are to one another as squares on the diameters.* The proof is an indirect proof in which the desired conclusion is reached by showing that assuming the ratio of the circles to be different from the ratio of the squares of the diameters leads to a contradiction.

The essential steps in the proof are shown on the next page with Figure 3 to illustrate them.

[5] *Ibid.*, vol. II, p. 191.

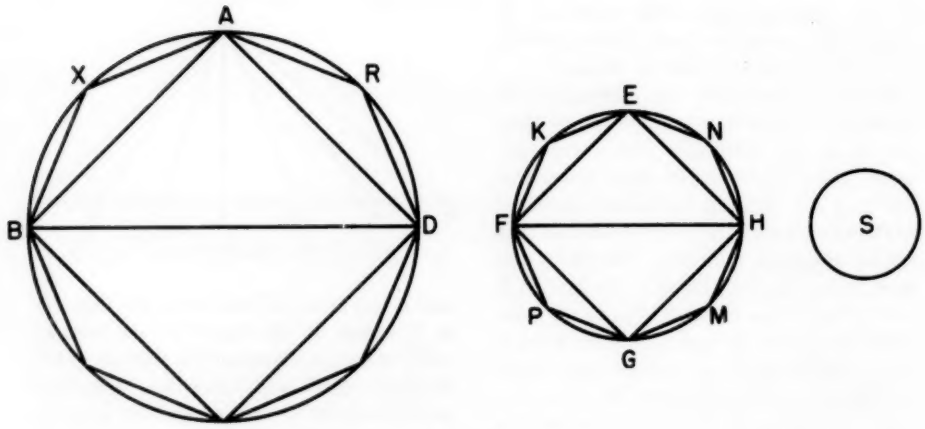


Figure 3

(1) If

$$\frac{\text{circle } BXARD}{\text{circle } FKENH} \neq \frac{BD^2}{FH^2}$$

then

$$\frac{BD^2}{FH^2} = \frac{\text{circle } BXARD}{S}$$

where S is some circle other than $FKENH$.

(2) S is either greater or less than circle $FKENH$. Assume it is less. (Here another type of indirect proof is being used. We will follow through only one-half of it, the "less than" part.)

(3) If $EFGH$ is a square inscribed in $FKENH$ it is more than one-half of $FKENH$ because the inscribed square is one-half of a circumscribed square and the circle is less than its circumscribed square.

(4) Use the midpoints $KNMP$ of the arcs as vertices of triangles based on the sides of the inscribed square. These triangles are together more than one-half of the area between the inscribed square and the circle.

(5) By repeatedly inscribing triangles in smaller and smaller circular arcs one can eventually arrive at an inscribed polygon which differs in area from circle $FKENH$ by an amount less than the difference between circle $FKENH$ and S. In other words, there is a polygon in $FKENH$ which is greater than S.

(6) Now inscribe a similar polygon in $BXARD$. Then:

$$\frac{\text{area of polygon in } BXARD}{\text{area of polygon in } FKENH} = \frac{BD^2}{FH^2} = \frac{\text{circle } BXARD}{S}$$

(7) But circle $BXARD$ is greater than the polygon in $BXARD$, hence the proportion shows that S is greater than the polygon in $FKENH$.

But this conclusion contradicts the conclusion of step (5), and hence the assumption of step (2) that S is less than circle $FKENH$ is false.

A similar procedure would show that S cannot be larger than circle $FKENH$ and thus the original theorem is established.

Note that Eudoxus' "method of exhaustion" makes use of Euclid X, 1: *Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there*

will be left some magnitude which will be less than the lesser magnitude set out.[6] In XII, 2, the lesser magnitude was the difference between circle *FKENH* and *S*.

This proposition was also used to prove X, 2: *If when the less of two unequal magnitudes is continually subtracted in turn from the greater, that which is left over never measures the one before it, the magnitudes will be incommensurable.* This is a *general* theorem on incommensurables, and, like XII, 2, also has embodied in its proof the subtractive approach which we discussed earlier.

The subtractive approach coupled with X, 1 (which can be derived from what is today called the Axiom of Archimedes) comes close to modern limit ideas. In Greek times they were closely related to the philosophical-physical problem of whether matter or magnitude is infinitely divisible. Were we to tell the entire story of incommensurables and all the related questions of Greek mathematics, we would have to discuss Democritus (470?–370? B.C.), the first atomist who wrote two books on irrational lines, Zeno (c.450 B.C.) and his paradoxes, the ideas of Anaxagoras (c. 440 B.C.) on infinite divisibility, as well as Antiphon (c. 430 B.C.), Bryson (c. 450 B.C.), and Archytas (c. 400 B.C.), who had some conception of the method of exhaustion before Eudoxus.

However, we will close our discussion of incommensurables through the Greek period by quoting B. L. van der Waerden on a topic also related to some studies of Greek art. Some persons have termed the Greeks "tactile" minded, emphasizing sculpture more than painting, for example, and have cited the geometric nature of Greek algebra as further evidence of this. Van der Waerden writes, however, "The Greeks knew irrational ratios very well—that they did not consider $\sqrt{2}$ a number was not ignorance, but a strict adherence to the definition of number. . . . In the domain of numbers [as viewed by the Greeks] $x^2=2$ can not be solved, not even in the ratios of numbers. But it is solvable in the domain of segments; indeed the diagonal of the unit square is a solution. . . . It is therefore logical necessity, not the mere delight in the visible, which compelled the Pythagoreans to transmute their algebra into a geometric form."[7]

[6] *Ibid.*, vol. III, p. 14.

[7] Van der Waerden, *op. cit.*, p. 125 ff.

"Pupils who have not an unusually strong natural bent towards mathematics are led to hate the subject by two shortcomings on the part of their teachers. The first is that mathematics is not exhibited as the basis of all our scientific knowledge, both theoretical and practical: the pupil is not convincingly shown that what we can understand of the world, and what we can do with machines, we can understand and do in virtue of mathematics. The second defect is that the difficulties are not approached gradually, as they should be, and are not minimized by being connected with easily apprehended central principles, so that the edifice of mathematics is made to look like a collection of detached hovels rather than a single temple embodying a unitary plan."—*Taken from* The Common Sense of the Exact Sciences *by William Kingdon Clifford. Pp. vii–viii.*

• MATHEMATICAL MISCELLANEA

Edited by Adrian Struyk, Clifton High School, Clifton, New Jersey

Three folding models of polyhedra

by Adrian Struyk

Folding models of the sort considered in this article usually arouse considerable interest. They are designed particularly for showing how one polyhedron may be inscribed in another and for aiding the investigation of relationships involved. Patterns for three of the simplest of such models are presented below—one for a cube with inscribed regular tetrahedron, a second for a rhombic dodecahedron with inscribed cube, and the third for a rhombic dodecahedron with inscribed regular octahedron.

Let A be a particular vertex of a cube. On each of the three faces which meet at A draw the diagonal from A. Let these be AB, AC, AD. On the remaining faces of the cube draw BC, CD, DB. These also are diagonals of the respective faces. The six named diagonals are the edges of a regular tetrahedron inscribed in the cube. Cutting along the faces of this inscribed tetrahedron frees it from four other tetrahedra —four "corners"—which, in effect, surround it to make up the cube. The model to be described produces just this effect.

Figure 1 consists of an equilateral triangle surrounded by isosceles right triangles and furnishes a pattern for one corner of a cube. Construct four of these tetrahedra, and paste them, equilateral face down, on a pattern for a regular tetrahedron. The result looks like Figure 2. Bring the vertices X, Y, Z of this model together at one point by folding along RS, ST, TR. This forms a cube with a regular tetrahedron inscribed. Unfolded,

Figure 1

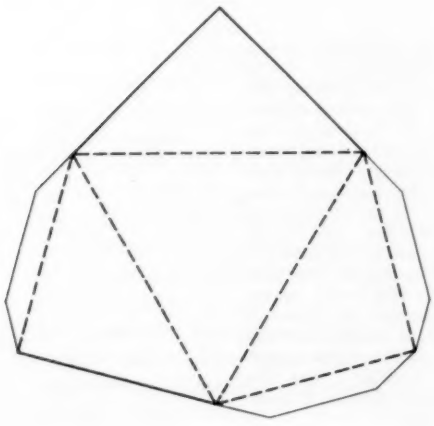
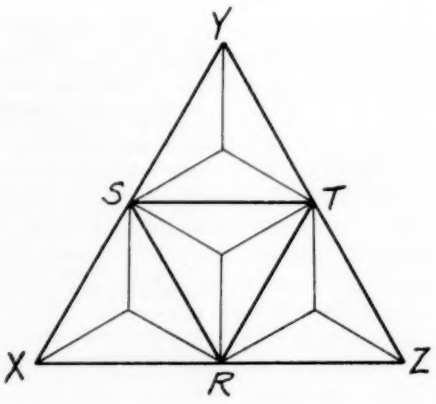


Figure 2



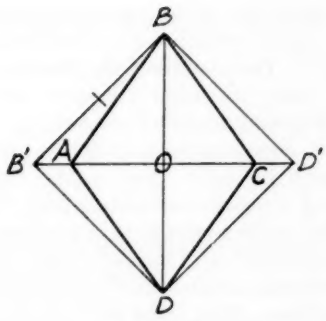


Figure 3

the model presents an intriguing appearance, and its purpose is not readily apparent. A very pretty effect is achieved if the model is painted with three colors so that opposite faces of the cube have the same color. Model airplane dope is excellent for this purpose.

The twelve faces of a rhombic dodecahedron are congruent rhombuses of such shape that the longer diagonal is $\sqrt{2}$ times the shorter one. A convenient way to lay out such a polygon is shown in Figure 3. $ABCD$ is the required rhombus, obtained by making OA and OC each equal to half a side of the square whose diagonals are BD and $B'D'$.

If on each face of a rhombic dodecahedron the shorter diagonal is drawn, the twelve line-segments are the edges of the inscribed cube. This cube is surrounded by six pyramids. For a model use Figure 4, a square with four adjoining triangles similar to triangle ABC of Figure 3. Construct six square pyramids according to this pattern, and paste them, square face down, on a pattern for a cube. The result looks like Figure 5. Folded so that the six squares form a cube, the surmounting pyramids form a rhombic dodecahedron.

The twelve line-segments obtained by drawing the longer diagonal on each face of a rhombic dodecahedron are the edges of a regular octahedron. On each face of this inscribed octahedron there is a triangular pyramid. The corresponding model, therefore, requires tetrahedra patterned after Figure 6—a regular triangle surrounded by three triangles similar to triangle ABD of Figure 3. Make eight such

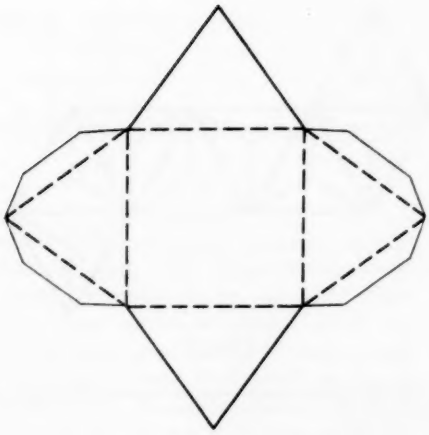


Figure 4

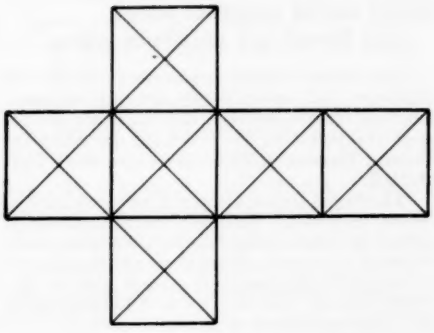


Figure 5

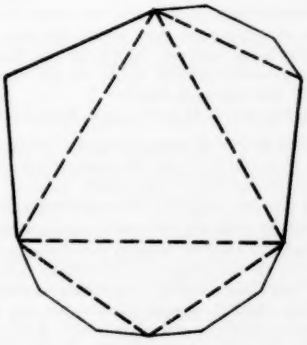


Figure 6

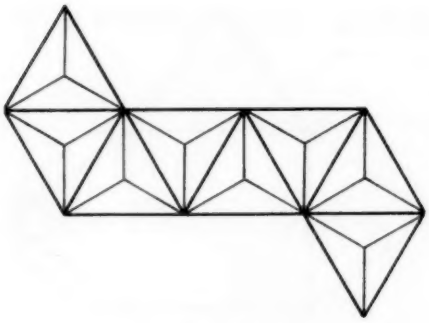


Figure 7

tetrahedra, and mount them on a pattern for a regular octahedron, as indicated in Figure 7. The rhombic dodecahedron appears when the eight triangles are arranged to form a regular octahedron.

Both dodecahedral models are greatly enhanced by color. Just as the twelve edges of a regular octahedron form three squares, so the twelve faces of the rhombic dodecahedron fall naturally into three square bands, or girdles, of four faces each. Using three colors, paint the four faces of a band with the same color.

Soviet aerial progress poses real threat, air chieftain warns

The United States needs more and better engineers and scientists to develop superior weapons for the nation's defense if we are to meet the threat of the U.S.S.R., according to General Thomas D. White, Air Force Vice Chief of Staff.

The Soviet Union, General White warned in a recent nationwide address, has come from a period in which most of the population was illiterate to a point where their present output of scientists and engineers exceeds that of the United States, and they are now showing signs of technological maturity.

Even though many of their spectacular successes have been the results of duplicating basic equipment taken from the Free World nations, the Soviets have accomplished many things that our experts said they could not do. Even more surprising, some of the things the Russians were supposedly unable to do at all, they not only did, but did in a hurry.

As examples of this, General White cited:

> The B-29—It was thought in this country that it would take the Reds six or seven years to duplicate it. It took them two.
>
> The jet engine—It was estimated it would be difficult to make a good copy of a British jet engine. Russia produced a better version of this engine in less than a year.
>
> The atom bomb—American experts predicted that it would take from six to ten years to produce a similar bomb, if they were able to produce it at all. The Soviets rocked the world with a nuclear explosion in about three years.

General White said that this knowledge is evidence of the Soviets and the advances which are being made by them.

America's answer to this growing Red threat lies in "Power for Peace," he said, and this power lies in having forces in being and in developing superior weapons systems which will keep the United States ahead of the enemy. — *Taken from* Planes, *official publication of the Aircraft Industries Association of America. September, 1955, Volume 11, Number 8.*

Plane-makers face weight problems

"Designers of America's sleek jet fighters and bombers have become as figure conscious as any of the famous couturiers—Dior, Ceil Chapman or Claire McCardell. And, even though for an entirely different purpose, they are far more weight-conscious of their aircraft models.

"The design engineer of one of this nation's largest aircraft manufacturers, for example, just recently ordered a streamlining of an electrical plug and receptacle. The redesigned items would weigh four tenths of an ounce instead of 3.2 ounces. With forty of the plugs to be used in a new jet fighter model, this would mean a total weight saving of 7.3 pounds.

"Considering that these accessories are to be installed aboard a 30,000-pound jet fighter plane, saving 7.3 pounds doesn't seem like much to worry about. But that small saving of weight in the design of an electrical unit resulted in a reduction of $3,650 in the final cost of that fighter. Multiplied in terms of the hundreds of planes on order, the saving on this one item passed on by the aircraft manufacturer to Uncle Sam's taxpayer becomes a figure to reckon with."—*Taken from* Planes, *Official Publication of the Aircraft Industries Association of America. November, 1955, Volume 11, No. 10.*

• MEMORABILIA MATHEMATICA

Edited by William L. Schaaf, Brooklyn College, Brooklyn, New York

Philately and mathematics—a further note

Alluding to the permanence of mathematical achievement, G. H. Hardy[1] wrote:

> In these days of conflict between ancient and modern studies, there must surely be something to be said for a study which did not begin with Pythagoras and will not end with Einstein, but is the oldest and the youngest of all.

By a quaint turn of poetic justice, both Pythagoras and Einstein have recently been honored on postage stamps, each for the first time, and only a few months apart. In the latter part of 1955, Greece at long last honored the Sage of Samos by observing the 2,500th anniversary of the founding of the first school of philosophy by Pythagoras at Tiganion on the island of Samos. To commemorate the occasion, the name of Tiganion was changed to Pythagoras, and a set of four stamps was issued: 2 drachma green, 3.50d black, 5d plum, and 6d light blue. The 2d and 5d stamps picture an ancient coin and his bust; the 6d stamp shows a stylized map of the island of Samos; and the 3.50d stamp, bearing the legend

ΠΙΤΘΑΓΟΡΕΙΟΝ ΘΕΩΡΗΜΑ

dramatically pictures the famous right-triangle-with-squares in black and gray against a white background. A very attractive set of stamps, in the humble opinion of an ardent collector.

Albert Einstein (1879–1955), the eminent theoretical physicist and mathematician whose contributions have rocked the very foundations of Western culture, was honored, appropriately enough, by the state of Israel, which issued a commemorative 350-pruta stamp in early 1956. The stamp carries a portrait of the renowned mathematician in his later years, and inscribed near the portrait in the vignette is the fabulous formula, $E=mC^2$.

To round out this little note on mathematics and philately, we cannot resist calling attention to a few miscellaneous stamps related to mathematics that have not been pointed out elsewhere so far as we know. There are at least three stamps showing the Trylon and Perisphere of New York's World's Fair days: *Iceland*, 20A., crimson, 1939 (#213); *Ecuador*, 2c., light olive green, 1939 (#388); *United States*, 3¢, deep purple, 1939 (#853).

A stamp incorporating a stylized spherical triangle in its design is *Iceland*, 35A., bright ultramarine, 1940 (#233); stamps featuring the sector of a circle are *Chile*, 40c., carmine and 2.50 p., deep blue, 1947 (#247, 248); and a stamp depicting a segmented world map is *Australia*, 3½d., gray and red, Y.M.C.A. centennial issue (1955). Two stamps showing a modern telescope are *United States*, 3¢, blue, Palomar Mountain Observatory issue, 1948 (#966), and *Japan*, 8y., dark blue-green, 1949 (#478), commemorating the 50th anniversary of the Mizusawa Latitudinal Observatory. Then there is the stamp showing the surveyor's triangulation point, *Switzerland*, 70c., purple, 1949 (#339); also the prize-winning stamp design showing a glider symbolizing aerodynamic buoyancy, *Switzerland*, 1.50 fr., dark violet and yellow, 1949 (#C 45). Finally, a 25-lire gray-green stamp of large format was issued by Italy late in 1955 to commemorate the *Centenary of*

[1] *A Mathematician's Apology* (Cambridge University Press, 1940), p. 16.

Professional Instruction in that country. The stamp shows a young student pointing to a drawing board which bears what appears to be a problem in geometry. In his left hand is a smaller drawing board and drawing instruments; to his right are a triangle and compass, an Etruscan vase, and a hammer and anvil.

Of course, some mathematically minded philatelists, or should we say philatelic-minded mathematics teachers, will also wish to include various older and current issues of United States stamps bearing portraits of Jefferson and Franklin—the former because of his interest in surveying and in systems of weights and measures, and the latter because of his association with the famous 8×8 magic square which bears his name. Including Jefferson and Franklin in such a collection is scarcely less justified than including Avicenna or Farabi; however, that is merely one opinion. More important, we have seen a stamp (one of a set of perhaps 6 or 8) from some European country (we have forgotten which country) which depicts a mathematics classroom with mathematical symbols on the blackboard. We even seem to remember that the stamp is rose, or carmine, or red. If any reader can identify this stamp, we should indeed be grateful. It is possible that *two* stamps of this series carry the same scene of a mathematics classroom.

The dawn of an era

By the middle of the 18th century the initial preparations were over. . . . Before more than a streak of grey had appeared at the horizon's rim, they proclaimed the dawn and announced how wonderful it was, how marvelous the new day would be.—*Lewis Mumford*

As Mark Napier once observed, "The century which commenced with the *Canon Mirificus Logarithmorum*, and was followed by the *Novum Organum Scientiarum*, deserved to be closed by the *Principia Mathematica;* and thus it is that Napier and Bacon and Newton created the transcendant era of science and, to use a congenial phrase, brought up gloriously the lee-way of old England."[2]

What in historical perspective is unquestionably a memorable and fascinating era in the development of mathematical thought, has been characterized in glowing terms by Montucla[3] in the following passages. For us they recapture the spirit of a bygone era with mellow nostalgia and genuine fidelity.

Among the ages which have successively contributed to the advancement of the sciences, that which is now fleeting away [the 18th century] holds undoubtedly the highest rank, and probably no succeeding age will deprive it of that elevation. Far be it from us to fix a limit to the human mind. Who knows the last boundary of knowledge, or where she must stay her step! Day after day uttereth knowledge, and to disregard the progress of discovery would be to withhold unjustly the tribute that is due to our illustrious contemporaries. Still, when we regard the wonderful flight which the sciences, and especially the mathematics, took in the *seventeenth* century, we must admit, that, whatever perfection they may receive from succeeding ages, a vast portion even of their glory will ever reflect back upon the age which so propitiously commenced the career. How brilliant is the spectacle which that era presents! How fascinating and admirable to the philosophic eye! If we turn to the pure mathematics, we find in the first years of that century LOGARITHMS, that invention so ingenious, and whose utility surpasses its ingenuity. We perceive the algebraic analysis, or the resolution of equations, greatly advanced by the discoveries of Harriot, Descartes, Newton, Halley. A new geometry, generated in the hands of Cavallieri and cultivated by others, aspires to researches far beyond the penetration of antiquity. Descartes in the meanwhile explores another path, and, applying the analysis to his geometry, gives the theory of curves an extension and play hitherto unknown and invents a variety of methods of solving with perfect certainty the most difficult problems in that branch of science. Fermat, the rival and contemporary of Descartes, pursues the same career and promulgates inventions in which the germ of the new calculus is greatly developed. Wallis, Barrow, Gregory, enrich geometry with a multitude of new methods and discoveries. Newton at length gives birth to that sublime geometry, compared with which the labours that paved the way were as trifles, and has furnished the only key to those difficult investigations which occupy the geometers and naturalists of

[2] Mark Napier, *Memoirs of John Napier of Merchiston* (Edinburgh: Wm. Blackwood, 1834), p. 374.

[3] *Histoire des Mathématiques* (1758), II, 2–4.

the present day. If we carry our view to the mixed mathematics, we will be no less delighted with the prospect of their acceleration. Mechanics present to us the laws of motion and its communication, the laws of the acceleration of heavy bodies, of the path of projectiles, of the motion and reciprocal action of fluids. We see them enriched by several profound theories—such as the centre of oscillation, the resistance of fluids, the doctrine of central forces, &c. At the same time the progress of optics is proceeding with equal brilliancy, the laws of vision and refraction unfolded, and a new science rises from that foundation. The telescope and microscope afford aids unknown to antiquity, the rainbow is submitted to reason, light is analyzed, and the various refrangibility of colours detected, the reflecting telescope is conceived and constructed with success, astronomy, in fine, presents us at once with the discovery of the actual forms of the planetary orbits and of the laws which preside over the celestial motions. Soon after the invention of the telescope, we see astronomers as it were soaring into space, descrying the spots on the sun, the motion of that luminary round its axis, the phases of Venus and Mercury, the little planets which attend like moons the steps of Jupiter and of Saturn with his marvellous ring—phenomena which shed a meridian light upon the true system of the universe. Upon these observations geography is entirely remodelled, the earth is measured with an accuracy far surpassing the measurement of the ancients, and her true form is ascertained, the truth of Kepler's observations is demonstrated by means of a profound application of geometry and mechanics to the motion of the heavenly bodies, the comet is controlled into a planet, and the career of that rare apparition submitted to calculation. The moon, so long rebellious to astronomy, is captive at length, and her eccentricities accounted for. And at last, from the hands of the immortal Newton, we receive the system of physical astronomy, the masterpiece of geometry and mechanics, accumulating daily new confirmation from the combined labours of geometricians and observers.

On the scope of mathematics—then and now

Still steeped in an aura of lavender and old lace suggested by the above quotation from Montucla, we cannot resist reprinting an excerpt from Thomas Thomson's *History of the Royal Society*.[4] The passage appears at the beginning of the section on "Mathematics" and alludes briefly to the meaning and scope of mathematics as regarded by scholars a century and a half ago.

[4] (London: 1812), p. 239.

The word *Mathematics*, originally Μαθησις, signifies, literally, *instruction;* and is usually supposed to have been applied by the Greeks to a science of which they were in some measure the inventors, in consequence of the high estimation in which they held it. This science treats of those *quantities* which are capable of being measured, and which on that account have been called *proper quantities*. They consist of the following three, namely, *extension, duration, number*. All other quantities are *improper*, and can only be introduced into mathematics, by artificially applying to them some proper quantity as a measure. Thus *velocity*, considered by itself, is not susceptible of measurement, and, of course, we cannot determine how much one velocity surpasses another. But if we make the *space* passed over in a given time the measure of velocities, then they become susceptible of accurate comparison, and consequently of mathematical investigation. Some improper quantities are of such a kind that we cannot apply any proper quantity as an artificial measure of them. Beauty, virtue, love, and many others, are of this nature. Such quantities can never become the objects of mathematical investigation. We may, indeed, throw our reasoning respecting them into a mathematical shape, and ring changes upon mathematical formulas; but such reasoning can never promote our knowledge of these improper quantities, or contribute in the least to the advancement of truth.

The proper subjects of mathematics, then, are *number* and *extension;* for *duration* may be easily joined to either the one or the other of these quantities. Hence the science naturally divides itself into two parts; namely, *arithmetic*, which treats of *number;* and *geometry*, which treats of *extension*. These two parts were, for many ages, considered separately. But, they have been gradually so united and blended together, that, in the present improved state of the science, they no longer admit of separate investigation. They constitute two streams, rising from distant sources, and running a long and parallel course: at first, only small and inconsiderable, but, fed each during its progress by a thousand rills, they swell into two mighty rivers, which, gradually approaching, at last mix their waters together, and flow united into the vast ocean of truth.

Mathematics constitute the most splendid monument of the human intellect. They are of vast extent, and, unlike the other sciences, they have never suffered any retrogression. Every mathematical truth, when once developed by that rigid principle of demonstration, which alone the true mathematician admits, constitutes ever after a part of the science, which no man, who understands the subject, will venture to call in question. At times, indeed, the progress of the science has been in a great measure stationary, while at others it has moved with prodigious celerity but at no period has it been retrograde. Mathematics have been divided into two great branches. The first branch, which

treats of pure number and extension, has been called *pure mathematics;* the second branch comprehends most of the sciences belonging to *mechanical philosophy;* which, by means of certain contrivances, have been subjected to mathematical investigation, and owe most of their improvements to that powerful auxiliary. This branch, which is of prodigious extent, has received the name of *mixed mathematics.*

How naive all this sounds today! What incredible advances in mathematics have been made since then! How utterly changed, in the span of scarcely four or five generations, have become our concepts of the very nature of mathematics. One is tempted to speculate on what sort of an appraisal a future biographer will shortly make of the viscissitudes of science during the fifty years we have just left behind and how that description will sound to a reader a century and half hence!

An "apology" for mathematics

In a facetious manner of speaking there are doubtless many persons who, with some feeling in the matter, are quite convinced that mathematics has much to "apologize" for. More seriously, however, it may be said that mathematics needs no apologies. Indeed, this is the theme of one of the most delightful and charming little volumes about mathematics ever written. The late G. H. Hardy was a British mathematician of international eminence; the essay is one of rare grace and lucidity.[5]

It should be understood that Hardy makes a clear distinction between the "real" mathematics and the "trivial" mathematics, or, as we would say, between "pure" mathematics and "applied" mathematics. In referring to applied mathematics as trivial, there is no intention to belittle or detract from the "practical values" and services rendered by mathematics to science, engineering, and technology. It is simply that Hardy is not talking about *that* kind of mathematics. Its uses are obvious. He simply wishes to defend the existence of the other kind: "A man who sets out to justify his existence and his activities has to distinguish two different questions. The first is whether the work which he does is worth doing; and the second is why he does it (whatever its value may be)."

In offering his "apology" for the career of a pure mathematician, Hardy answers the second of these questions first. If he is worthy of being called such, the mathematician pursues his activities because that is the thing he can do best, because he has a real talent for it, and because it yields unique satisfactions, especially if he can do it really well.

The heart of the apology is perhaps to be found in the answer to the first question. The crux of the matter lies in the very nature of pure mathematics, which is characterized by three outstanding qualities: (1) its *permanence;* (2) its *beauty;* and (3) its *seriousness.*

What the mathematician does may be small, but it has a certain character of permanence. "In these days of conflict between ancient and modern studies, there must surely be something to be said for a study which did not begin with Pythagoras, and will not end with Einstein, but is the oldest and youngest of all. . . . Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not."

The beauty of mathematical theorems and their demonstrations, according to Hardy, lies in the quality of *unexpectedness* combined with *inevitability* and *economy.* "The arguments take so odd and surprising a form; the weapons used seem so childishly simple when compared with the far-reaching results; but there is no escape from the conclusions."

Real mathematics is important. The best mathematics is serious as well as beautiful. A serious theorem is one which deals with *significant* mathematical ideas. To illustrate what he means by this, Hardy discusses two well-known theorems

[5] G. H. Hardy, *A Mathematician's Apology* (Cambridge University Press, 1940), pp. 6, 50.

—Euclid's proof that there exist an infinite number of prime numbers, and Pythagoras' proof that $\sqrt{2}$ is irrational. We shall not deprive the reader of genuine pleasure by giving the proofs here. But these proofs make perfectly clear the meaning of *significant ideas*. A mathematical idea is significant when it possesses both *generality* and *depth*. Here, generality does not mean a generalization in the ordinary sense—all mathematical theorems are generalizations in this sense. The idea should be general in the sense that it is a constituent in many mathematical patterns, that it can be used in the proofs of many different kinds of theorems, that it admits of extension and is typical of a whole class of theorems, and that it reveals interrelations among other mathematical ideas. The quality of depth has to do with difficulty, but is by no means synonomous with it. "It seems that mathematical ideas are arranged somehow in strata, the ideas in each stratum being linked by a complex of relations both among themselves and with those above and below. The lower the stratum, the deeper (and in general the more difficult) the idea. Thus the idea of an 'irrational' is deeper than that of an integer; and Pythagoras' theorem is, for that reason, deeper than Euclid's."

On the calculus as a required subject

Every so often mathematics teachers are sharply reminded in one way or another of the "disciplinary values" of the subject they teach. Recently a most intriguing note along these lines appeared from the pen of a free lance writer and former university teacher of journalism, Mr. John W. McReynolds, entitled: "An Enquiry, Whether the Differential and Integral Calculus Should be Required in the Undergraduate English Curriculum."[6]

[6] *Bulletin, American Association of University Professors* (1955), vol. 41, no. 2, pp. 351–358.

We wish that there were space to reproduce the entire delightful paper, which is in the form of a "letter to the writer's Cousin, a man who has long professed English literature, from J. W. McR., farmer, of Pottawatomie County, Kansas." We must content ourselves with two brief excerpts reprinted here by courtesy of the editors of the *Bulletin*. The first selection tells its own story:

> . . . . I shall urge the calculus, as you will see, largely on the grounds that it enhances literacy.
>
> Language—and that this may be opinion rather than fact in no way alters the tenacity with which I hold to the opinion—begins in poetry and ends in the symbols of the mathematician. The general quadratic equation and its solution (for this much you may remember of your own schoolboy days) express, in a few symbols properly arranged and in a calm precision, what a host of untoward words could scarcely convey. The mathematician, as the novelist, spends years in the poignant examination of his own chosen aspect of the larger reality. When he has done, as the novelist, he wishes—indeed, he *must*!—communicate. It is more than mere conjunction of figures to observe that Dawn's rosy fingers and Einstein's e's and m's and c's are all of a piece.
>
> And now the drift of my first argument is at once made plain. A literate I take to be one who has some mastery of expression. Of absolute necessity in any mathematical enquiry is maximum, if not perfect, precision of expression. This perfection the mathematicians call "rigor." The conclusion follows naturally, and I do not intend to belabor the argument. It is probably within your own lifetime that the mathematics and classics departments anxiously embraced one another and, in an alliance born of desperation, suffered themselves to be called mental disciplines and allowed themselves to be pursued for this end in order to be pursued at all. How many poor unfortunates, desiring first to be loved for themselves alone and finding none to love them, have hopefully given up their womanhood for mere femaleness, and given it in vain!
>
> To continue, it is difficult to draw the line between weakness and perversion, and I shall make no attempt to declare whether my argument in favor of the calculus be weak or perverted. I simply make it. It has been made before. Mathematics should be required of the English undergraduate because it compels him to examine precise expression.

In the next passage, our champion is moved by consideration of the futility of understanding the true nature and sig-

nificance of modern science without at least some familiarity with the calculus. But let him speak for himself:

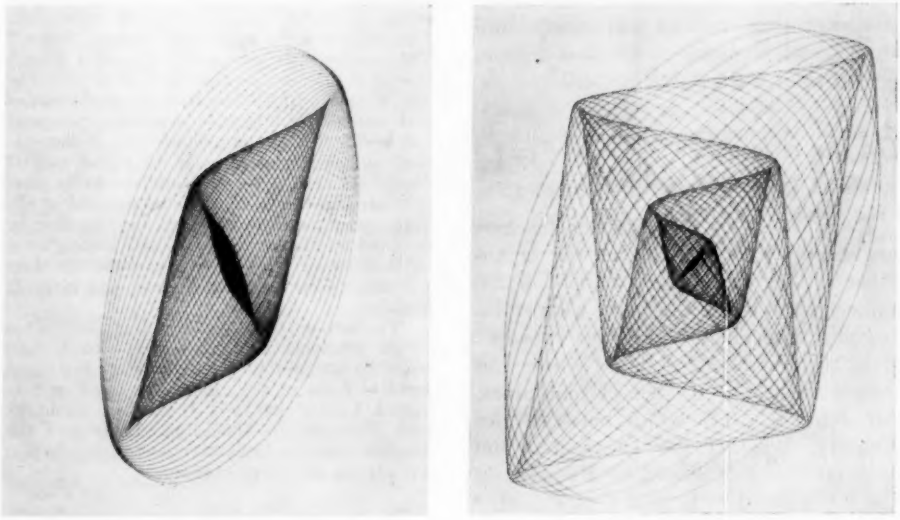
Algebra and plane geometry and the physics of a generation ago are all that you have of the inner workings of this vast intellectual, even cultural, stride called Science. I grant—Cousin, I urge—that in your own educated humanity you are aware of its effects; the man who is not must be a simple clod, indeed. You know what they have done, *but you know not how they have done it.* I suppose you have attained to such proficiency in the classics as had Leacock—you will remember that he could take a page of Latin and a page of Greek and tell at a glance which was which. I cannot, then, say that the language of science is as Greek to you—there is no comparison to make. You do not know it. . . .

It becomes my present task to demonstrate to you, clearly and conclusively, that the differential and integral calculus are the introductions to the language of science, that nothing less will suffice. My first argument, of course, is that I know more about it than you do; which is little enough, at that. I have sat in those classrooms longer and more recently than you. But I doubt me that you are presently prepared to receive a philosophy (as it were) from my hands, and I assume you wish a higher authority than mine. Such authority is, unfortunately, difficult to find. F. Klien in a technical work—he was a technical man—touches upon it:

. . . Everyone who understands the subject will agree that even the basis on which the scientific explanation of nature rests is intelligible only to those who have learned at least the elements of the differential and integral calculus. . . .

But he does not prove it, nor does he attempt to. The chairman of a friendly mathematics department tells me that no authority has bothered with such an exposition because for two centuries it has been an obvious, patent, almost a mere, fact!

Pendulum patterns



Pendulum Patterns, a student project by John Kennedy, Senior High School, Richmond, Indiana

A column of unofficial comment

The White House Conference on Education and its implication for mathematics teachers

Kenneth E. Brown, U. S. Office of Education, Washington, D. C.

The White House Conference on Education reflected the decision of the American people to have the best education obtainable for all American children. They are willing to pay for it and work for it. They are determined to have good schools, not only because of high ideals, but for survival.

The White House Conference on Education was the culmination of the active participation of over 500,000 men and women studying the problems of educating the youth of a democracy. Never in the history of civilization have so many people devoted themselves to such a concerted effort to improve education. The one opinion that was held by all participants was aptly expressed by the President of the United States in his opening address when he said: "Everybody must have a good education." No provision was made in their thinking for the illiterate common laborer. The atomic age is challenging our nation to educate its youth to their maximum capacity, and our citizenry is accepting the challenge.

What is good education?

There were differences of opinion as to what constitutes a *good* education. Dr. James R. Killian, Jr., Chairman, Subcommittee on "What Should Our Schools Accomplish?" stated in an address at a general session of the Conference that he had received many letters from individuals and organizations urging special emphasis on various areas of learning, but through all this diversity of views ran a conviction that our educational system "is not nearly good enough for our youth and our society."

As lay persons continue an active participation in improving education, each course must justify its existence in the curriculum. Mr. Killian was following this thesis when he reminded the Conference: "The question is not whether this course or that course of instruction has any good in it . . . for there are limits to what the schools can attempt." He continued his emphasis on priority of content by saying: "The average student has only a few pennies of time to spend on education, and he can't buy everything in the store. It is up to his elders to help him spend his time as wisely as possible—to purchase not just a lot of little educational trinkets, but something which will sustain him all his life."

What priority should mathematics have?

When it comes to listing educational priorities, where will mathematics stand? Can we rely on mathematics prestige of bygone days? Are our mathematics courses

preparing the engineer and scientist for his work? A large university reports that this year one-third of their Freshmen who desired to pursue scientific studies found it necessary to take high-school mathematics in college. Does the content of the courses in mathematics contain deadwood, or has it been brought up to date? Can we expect the mathematics of 30 years ago still to hold top priority in our schools? The pupil's time is limited. How much time should we ask him to devote to plane and solid geometry and numerical trigonometry? How high a priority does factoring and simplification of radicals command? We should have data on these and kindred questions. It is not enough to say that mathematics is an important part of our social heritage, or even that it is useful. Most knowledge is useful. We must now justify curriculum content in terms of degree of importance.

The final report of the work groups of the White House Conference on Education agreed that schools should teach arithmetic and mathematical skills, including problem solving and the ability to think and evaluate constructively and creatively. They further stated that improvement in our schools is desirable and necessary. Is not now an appropriate time to re-study our high-school mathematics program?

Teacher shortage and its results

A shortage of mathematics teachers is imminent. Last fall the average junior-high-school class in algebra contained 30 pupils—too many for individual instruction. However, if we are to maintain our present teacher-pupil ratio, the staffs must be enlarged more in the next 10 years than they have been in the past 35 years. According to the report, "Teachers for Tomorrow," by the Fund for the Advancement of Education, 51.2 per cent of all college graduates in the next 10 years will be needed in our schools. (This bulletin may be secured free by writing directly to 655 Madison Avenue, New York 21, New York.) Last year only one-fifth of the college graduates entered teaching. It is unrealistic to think that half of our college graduates will enter teaching. A shortage of mathematics teachers seems inevitable.

What are the possible results? One result may be the elimination of mathematics classes that do not have high educational priority. Should not mathematics teachers examine the courses to see (1) that only vital material is included and (2) that the ideas lead to modern advanced mathematics and their applications. For example, some pupils become rather proficient in factoring, but never gain the most elementary concepts in statistics—a most used area of mathematical concepts. Other pupils spend months memorizing data about triangles of an imaginary flat plane, but never receive a moments instruction about the geometry of the curved surface of the world on which they live.

Another result of a shortage of teachers is an increase in class size. Have we explored all ways of providing for the pupils in large classes? Physicians transfer many duties to technicians, nurses use nurses' aides, and engineers use electronic computers to increase output. Have we studied our classroom tasks to determine those that could be done by teachers' aides, the instruction that could be done by television, or the teaching that can be done better in large groups and that which is successful only in small groups?

Some studies indicate that teachers are obliged to use much time on clerical and other non-professional tasks. Elementary classes of 45 to 50 pupils with a teacher plus an aide are reported to have made greater progress than classes of 30 with an equally good teacher, but no aide. Should the plan be explored in the area of mathematics instruction?

Summary

The citizens of the United States want a good education for all pupils. They want a better education than is now provided.

The White House Conference on Education recommended the study of mathematics and an improvement in the teaching of these skills. Among the road blocks to the realization of these goals are the shortage of teachers and an overcrowded curriculum.

What are the implications of the White House Conference on Education for mathematics teachers? Mathematics teachers will be asked to justify the high priority of mathematics among other important areas of learning in the curriculum. High-school mathematics courses must be evaluated in terms of modern advances in applied mathematics and science. Someone may ask, "With all our scientific advancement, cannot more pupils be taught per teacher now than 50 years ago?" If mathematics is to hold its rightful place in education, we must have the answer to these and similar questions. Our house must be put in order without delay.

Letters to the editor

Dear Sir:

Various comments that have reached me about recent book reviews in THE MATHEMATICS TEACHER, plus Phillip Jones' letter to the editor (page 406) in the October issue of that magazine, plus Rosskopf's article on "What makes a good textbook review?" (pages 418–419) in the same issue, lead me to suggest a few comments as to the nature of a good review of any book.

Professor Dickson's article in the July-August (1924) issue of *The American Mathematical Monthly* suggested the problem of the real purposes of a good review. It complains, and justly so, of those who are lavish in their praise, when they review a book, without pointing out any of its defects; and also of those reviewers who seek "to convince the reader that the reviewer himself is a person who knows a great deal about the subject and how it should have been presented." These two types of reviewers may be characterized psychologically as those who are predisposed toward thoughtless acquiescence, or else those who seek and find only faults. If what I say here serves to hold each of these extreme types in check, the readers of current reviews will doubtless gain and be grateful. Moreover, the editors of certain current magazines will have less cause for worry and concern.

According to *The Saturday Review*, "The multiplicity of reviews serves as a public-opinion poll on which authors and publishers must depend. The reviewer's words of praise or enthusiastic approval form the basis of the structure erected by the author, the publisher, and the reviewer, for the author must be supported in his lonely and hazardous task by the publisher, and both must have the encouragement of the reviewer, whose final function it is to send the public into the bookstores. In the book-reviewing profession today, there seems to be no room for rancor or bad temper."

In general, this seems to be a fair statement of the purpose of a good book review. Now what are some of the detailed qualities of a good review that the reviewer should keep in mind? It seems to me that the purpose of a good book review should be to give a fair idea, as succinctly as possible, of

1. The general nature of the book, and the audience for whom it was written.
2. The purpose and scope of the book.
3. The general degree of scholarship shown in carrying out this purpose.
4. The literary ability of the author.
5. The general accuracy of detail.
6. The significant matters of weakness so that the reader may not have to rely too much on the author's statements.
7. Such mechanical features as the format, manufacture, illustrations, index tables, bibliography and typography.
8. Adaptability of the book to library needs.
9. The soul of the book.

If the above requirements are satisfied, book reviews will at least be more gratefully received by authors and readers alike.

W. D. REEVE,
Teachers College, Columbia University

Reviews and evaluations

Edited by Richard D. Crumley, University of South Carolina, Columbia, South Carolina, and Roderick C. McLennan, Arlington Heights Township High School, Arlington Heights, Illinois

BOOKS

College Algebra and Plane Trigonometry, Abraham Spitzbart and Ross H. Bardell, Cambridge 42, Massachusetts, Addison-Wesley Publishing Company, Inc. Cloth, xiii+408 pp., $4.50.

This text as the title indicates, combines the topics usually presented in plane trigonometry and in college algebra. A review of intermediate algebra is followed by a sequence of chapters which alternates, more or less, between topics in plane trigonometry and topics in college algebra. The book closes with a partial answer list and the usual four-place tables. There are extensive lists of exercises, mostly of a routine nature intended for drill. This reviewer found the pace of exposition rather rapid for this level.

Good points of the present text include: emphasis on the general angle and on the trigonometric *functions;* emphasis on inverse functions; a large number of exercises which apply algebraic skills to trigonometric functions; and three sets of review exercises each covering about one-third of the text.

Bad points of the present text (in this reviewer's opinion) include: too brief treatment of inequalities; inclusion of too many topics for a single course; treating determinants without mentioning matrices and as though they were useful for solving linear equations; use of limiting processes without adequate explanation in computation with the binomial series and in continuous compounding of interest; presentation of the usual semi-proof of the addition formulas for sine and cosine (a proof is, however, also presented); emphasis on reduction rules for the trigonometric functions.

Combinations of courses in elementary mathematics are quite popular at present in *colleges.* A combination of analytic geometry and calculus is particularly popular since calculus is such a powerful tool in analytic geometry, and because such combinations frequently omit just those topics of analytic geometry (such as the classification of conics) which many people nowadays feel should be treated by more advanced and more efficient methods (such as matric theory). The main reason for such combinations is, of course, the demand for early introduction of the methods of the calculus into the course in basic science and technology.

It is natural that attempts should be made to extend this idea to combinations of analytic geometry, plane trigonometry, and/or college algebra. It is the opinion of this reviewer that such combined courses will become really popular only when a number of rather special topics are deleted. Prime candidates for such deletion are the topics involving the proof of the binomial theorem by mathematical induction, systems of quadratics, determinants, and solution of oblique triangles by logarithms. The inclusion of all of these topics in the present text has made the basic task of the authors exceedingly difficult.

The sort of combination which this text represents seems somewhat out of place in high-school teaching. The high schools have more time to devote to these subjects. Nonetheless, there is great need for integration of the topics taught in high-school mathematics. There is greater need today for the integration of mathematics and physics in the high school than there was fifty years ago when the great mathematician, E. H. Moore, pleaded for it. Let us hope that the pressures for integration do not lead us to adopt a policy of integration by permutation of parts.—*M. F. Smiley, State University of Iowa.*

Mathematics and Measurements, Merrill Rassweiler and J. Merle Harris, Evanston, Illinois, Row, Peterson and Company, 1955. Cloth, ix+351 pp., $4.50.

This book is designed to develop skills in algebraic manipulation and problem-solving for people who will work with measurements. This objective is approached through presentation of the most frequently used mathematical skills together with techniques for using them. The presentation of these mathematical ideas is, however, extremely conservative in view of modern presentations of mathematics as a logical system, for there is very little explicit use of the logical structure of the number system or of elementary algebra. In fact, in several places, the reader's understanding of the role of postulates and of deductive proof is led astray. For example, in a section on calculations with signed

numbers, the reader is left with the impression that he has helped "deduce some rules for doing arithmetic with positive and negative numbers" (p. 42). What he has done is to observe several situations that serve as bases for *postulating* these rules. The commutative law is mentioned as important in adding groups of more than two signed numbers, but the associative law is not mentioned. In fact, on page 46 an example finding the product of four factors is attributed solely to the commutative law for multiplication, when it is the associative law that best explains the grouping of factors actually used there! With these exceptions the development of signed numbers is good in that it utilizes vector illustrations (without, unfortunately, calling them vectors).

The authors have done well in clarifying the much-abused concepts of exact, approximate, precise, and accurate measurements. Their discussion of errors is excellent and the principles of computing with numbers obtained by measurement are followed consistently throughout the book. But it seems unfortunate that they have chosen, in trying to clarify the concept of number, to use the adjectives "abstract," "concrete," "exact," and "approximate" to describe different kinds of numbers. Here, it seems, the reader will be not so much confused as misled into thinking that there are "approximate numbers" rather than "numbers that are approximations to a theoretical value."

A chapter on problem-solving has strength in its approach to problems with two unknowns. With no equivocation the authors plunge directly into systems of equations, using the substitution process exclusively. The chapter is weakened somewhat, however, by the use of a great many (too many) artificial mixture-, rate-, age-, and number-problems.

The text has an excellent supply of exercises. Indeed, the richness of the supply of different formulas is almost appalling. However, most of them are well identified with regard to meanings of symbols and area from which the formula was taken. Illustrations and diagrams, in the places where they are needed, have been chosen well and in good variety.—*Lyman Peck, Iowa State Teachers College, Cedar Falls, Iowa.*

Practical Mathematics Refresher, William D. Reeve and Clarence E. Tuites, New York, McGraw-Hill Book Company, Inc., 1955. Paper, viii+376 pp., $3.25.

This book presents a review of: (1) the fundamental arithmetic skills with integers and fractions; (2) informal geometry; (3) the simpler elements of algebra; and (4) scale drawings and numerical trigonometry. The book is organized as a combination textbook and workbook, with exercises at the end of each section. The exercises are printed on perforated pages so that they can be easily removed. The exercises comprise 212 pages in all. Two of the twenty-three sections deal with topics which might not be expected in a book of this kind, e.g., "Section 10: Measures of Central Tendency—Averages," and "Section 15: Navigation and Aeronautics."

It must be remembered that this book is intended as a refresher, so it is quite consistent for the explanations to be brief. The result, however, is more along the line of "how to do it" than "why we do it this way." "Section 13: Geometry of Size," is the best section, in the opinion of the reviewer. Many of the definitions describe rather than define, but only one definition is unconventional: "Factoring is the name given to the process of determining the prime factors of an algebraic expression" (p. 310). The reviewer found only one mathematical error: "Thus the prime factors of ab^2xy^2 are a, b^2, x, and y^2" (p. 310).

Even though answers are given to odd-numbered exercises, this book is not well suited for a person who wants to review independently the concepts and skills of elementary and secondary mathematics. The explanations are too cryptic for this purpose. This means that a teacher must be available in conjunction with the book. The implication of all this is that the book can be used best in college courses or in industrial courses or by an individual with a tutor.—*Richard D. Crumley, South Carolina University, Columbia, South Carolina.*

Shop Mathematics, Claude E. Stout, New York, John Wiley and Sons, Inc., 1955. Cloth, xi+282 pp., $3.70.

Shop Mathematics is organized to meet the needs of machinists, tool and die makers, and pattern makers. It incorporates the principles of arithmetic, algebra, geometry, trigonometry, logarithms, and the slide rule that are necessary in shop work.

The reviewer sought to analyze *Shop Mathematics* from the viewpoint of the shop employee, whose knowledge of mathematics need not be as complete or idealistic as is the case of those who study college-preparatory mathematics. The book will undoubtedly be beneficial to the user, for its applications are pertinent and numerous, and explanations are succinct. The mathematician might be offended by a statement on page 126 that will not affect most shop students. The title, "Pictorial Review of Point-Line-Angle Definitions," seems to be a misnomer. Is it possible for a definition to be pictorial, or is it the term that is described by the definition that is pictorial?

High-school mathematics teachers will find that a book of this type is a desirable contribution to their collection of reference books. Problems that illustrate applications will stimulate the learning of those general-mathematics students who wish to enter the skilled trades rather than a college or university. Plane-geometry students would be surprised to find the number of shop situations that require a knowledge of

geometry, and algebra students would be impressed by the amount of algebra that must be understood by tradesmen.—*Roderick C. McLennan, Arlington Heights High School, Arlington Heights, Illinois.*

FILMSTRIPS

Man and Measures: "Early Counting," "Early Measuring," "Early Time Telling," and "Geometric Figures," The Filmstrip House, 15 West 46th Street, New York 36, New York. 35 mm., 4-color filmstrips written and illustrated by Agnes Herbert, edited by Reid Irving; $6.00 each, or $20.00 for set.

Geared to the junior-high- and senior-high-school level, these filmstrips provide some of the history of the development of mathematics. "Early Counting" points out the need for number symbols and shows some of the early symbols used by man. "Early Measuring" describes some of the earliest units used in measuring and the way in which they became standardized. "Early Time Telling" surveys the various systems and devices used through the ages for measuring time. The last strip, "Geometric Figures," describes and illustrates the common figures in one, two, and three dimensions.

The two strips, "Early Measuring" and "Early Time Telling," are both very interesting and informative. Either one can be used effectively to bring about a fuller understanding of the struggles man has faced and overcome in the measurement of distance and time. The medium of filmstrip is used to good advantage for this purpose.

The other two strips, "Early Counting" and "Geometric Figures," are neither as interesting nor as successful as the two just discussed. "Early Counting" really deals more with early symbols for numbers than with counting. More serious, however, is the view taken toward the various systems of symbolizing numbers. Emphasis is given to dividing a unit into tenths, twelfths, and sixtieths rather than the grouping of ten, twelve, or sixty units to make a larger unit. "Geometric Figures" is criticized more severely because it seems out of place in this series. Furthermore, the geometric figures considered are described, rather than defined. The example shown for a triangular prism is unfortunate: a wedge-shaped piece of cake.

The photographic quality of all four strips is excellent, as is the quality of the illustrations.—*Richard D. Crumley.*

Have you read?

Guggenbuhl, Laura. "Karl Wilhelm Feuerbach, Mathematician," *The Scientific Monthly*, vol. 81, August 1955, pp. 71–76.

In my teaching I have always found that the students are interested in the people who developed the mathematics they are studying. Here is one I did not know: the "nine point circle" for many years was called the "Feuerbach Circle." It seems Karl Feuerbach first proved, at the age of 21, the theorem: "The circle through the feet of the altitudes of a triangle is tangent to the inscribed and escribed circles of the triangle." This article gives a brief story of his short, but unconventional life. Your students will be interested not only in his mathematical contributions to the nine-point circle problem, projectors, and co-ordinate geometry, but also in Feuerbach's political activities, his mental quirks, and why he is so little known in the history of mathematics.

Hess, Adrien L. "The Place of Geometric Construction in Plane-Geometry," *School Science and Mathematics*, November 1955, pp. 594–98.

Shall we or shall we not consider the straightedge and compasses as the only allowable instruments for geometric construction?

Mr. Hess has gathered together the opinions of 131 teachers, 14 textbooks, and 20 achievement tests on this question. Many revealing comments are listed. In most cases, textbooks gave no reasons for the restriction, and only one did not impose it. Out of 3,109 points on the tests, 168 were allocated to constructions. Seventy-one per cent of the teachers agree with the restrictions, and you will be surprised and interested in their reasons. Conclusions are hard to draw, so you should read the article and draw your own.—Philip Peak, *Indiana University, Bloomington, Indiana*

• TIPS FOR BEGINNERS

Edited by Francis G. Lankford, Jr., Longwood College, Farmville, Virginia

Pupil discovery in junior high school mathematics

William C. Lowry, University of Virginia, Charlottesville, Virginia

Some teachers feel that the method of pupil discovery in teaching mathematics is too slow, too time consuming. Often, however, when the teacher merely "covers" the material of a course, it is found that what he thinks has been taught remains hazy and only partially revealed to the pupil. On the other hand, when a pupil discovers for himself the concepts and processes of mathematics, he not only is likely to retain them longer, but he has learned a method by which he may rediscover or redevelop them on his own. Moreover, if the method used in the classroom helps a pupil to learn on his own, the teacher will have helped him acquire a power to continue learning long after the classroom has been left behind—a power that a lot of unrelated facts cannot give him.

The junior high school is generally considered an excellent place to emphasize measurement. Many of the relationships of geometry present excellent opportunities for discovery through the use of measurement. Here the discovery method has the student make the required measurements in a variety of cases. He records these measurements in a table. He may need to derive additional data by performing various mathematical operations on the original data. Then, through a study of his tabulated data, he is helped to discover the relationship involved and to make the appropriate generalization.

We may take the following as an example of a discovery we would like pupils to make through the use of measurement: *An angle inscribed in a circle is equal in degrees to one-half its intercepted arc.* The pupils are directed to draw some unequal circles and to inscribe two or three different sized angles in each circle. Each pupil measures in degrees both the inscribed angle and its intercepted arc for each of the several cases and lists these pairs of readings in a table something like the following:

Case	Number of degrees in the inscribed angle	Number of degrees in the intercepted arc
I	22	43
II	46	90
III	15	30
IV	71	140
V	103	205

Since the relationship desired may be seen from a study of the original measurements in this case, the pupil does not need other columns of computed or derived data. Once the pupil thinks he sees the relationship involved, he should be directed to try a few more cases to substantiate further his conclusion. It is then considered good practice to have the child write a statement of his generalization in his own words.

This last step—writing a statement of his generalization—is often a very difficult one for the pupil. The child may well be telling the truth when he says, "I know it, but I can't write what I want to say." The purpose in having him write the generalization in his own words is not to test whether or not he has discovered the proper relationship and understands it; that test can and should be made primarily by seeing if he can use the generalization correctly.

We have the child write his generalization in his own words because such a statement is a summary of his thinking and serves to strengthen his grasp of the relationship. But there is an additional reason. A very important part of a teacher's job is to help the child learn to express himself. Teachers in mathematics have a special opportunity to help the pupil learn to express himself clearly and concisely. A laboratory approach in mathematics gives him something to write about that he himself has thought through and developed. The pupil's statement differs in this respect from some of his writings in which he reads and then simply rearranges the words and thoughts of others.

The foregoing discussion does indicate that the method of discovery is relatively slow. It is time consuming for children to make tables, to draw figures, to measure carefully, and to record these data. We have no way of knowing just how long it may take a particular child to see a relationship of mathematics. The writing and correcting of the generalization is often a long and laborious task for a junior-high-school pupil. The following paragraphs suggest some practices which will aid in shortening the total process. I know of no way to hasten the thinking the child must do for himself.

1. Use previously prepared forms on which to record the data. These forms may be dittoed or mimeographed on relatively inexpensive paper. The pupil has only to fill in the title and the appropriate headings for the columns. A space should be left at the bottom of the form where he may write his generalization.
2. If the problem is one which the entire class is to consider at the same time, whenever practicable it should be of such a length that the data may be collected, the desired relationship found, and some discussion of the problem made during one class period. If the problem must be carried over to the next day, a certain amount of time is lost in getting back into the channel of thinking. It will be necessary to observe the work of many pupils, in particular the ones known to be slower, and to give additional hints and suggestions to them.
3. If the problem is of such a nature that the total procedure is too long for one class period, it may be handled in one of several ways:
   a. Give directions concerning what to do, what data to collect, and the kind of relationship sought. Let the pupils collect the data and proceed with the generalization as far as possible at home.
   b. Let the pupils collect the data during a class period, and ask them to look for and state the generalization as homework.
   c. When the process of collecting the data itself is long and involved, it is advantageous to have the class work co-operatively to supply the data. Thus each pupil may make the necessary drawings and measurements for one or two cases. Then, data from the various members of the class are listed on a master table drawn upon the blackboard. Each pupil may study the composite master table or copy the data from that table to his own, and use it to try to make the discovery.
4. Many generalizations are arrived at rather easily; for example, *the angles opposite the equal sides of an isosceles triangle are equal.* After looking at two or

three isosceles triangles, the child "sees" that the base angles are equal. Even though he can produce no deductive proof, he would be surprised if they turned out unequal. Have him test a few more cases, but do not insult his intelligence and use time unnecessarily by requiring him to measure the base angles of 15 or 20 different isosceles triangles before he is permitted to make a generalization. In fact, establish early the idea that as soon as he thinks he sees a relationship, he should start testing to see if each succeeding case agrees with his thinking.

5. To help pupils in the discovery of concepts and processes of mathematics there are many commercially-produced devices, besides those that can be made by the teacher and pupils. Such a laboratory of equipment may be built up over a period of years. The pupils also need some basic measuring and drawing instruments such as rulers, protractors, and compasses. It is very helpful to have T-squares and drawing triangles. While the junior high school pupil should learn to make certain of basic constructions with straightedge and compasses, he should by no means be limited to these. Permit him to use T-squares, drawing triangles, and measuring and drawing tools of all kinds to "draw" his figures according to the given conditions. Such drawings are quicker to make than traditional constructions. The completed figure is neater and has fewer distracting auxiliary lines. Moreover, the junior high school pupil is too young to appreciate a theoretical geometry involving the limitations on instruments imposed by Plato and Euclid.
6. The pupil-discovery method should be inherent in the teacher's way of teaching. The pupil must be helped to discover mathematical relationships again and again in this manner if it is to become a part of the way he attacks problems. At first the teacher will need to give rather explicit information on what figures to draw, what measurements to make, and the kind of relationship sought. Later, as pupils become more familiar with the discovery method, the teacher may simply suggest the figure and the relationship involved.
7. While it is important that the inductive method be used often if the pupil is to learn it as a method of attacking new problems, its detailed use should not be insisted upon where a bit of deductive reasoning is just as easily used and just as convincing for the student. For example, once the child knows that the sum of the interior angles of a plane triangle is 180°, which he may discover through measurement, he can then find the sum of the interior angles of any plane polygon without recourse to further measurement. He divides the polygon into triangles by drawing as many diagonals as possible from any one vertex and reasons that the total number of degrees in the interior angles of the polygon will be 180° multiplied by the number of triangles thus formed.

Finally, I would cite my experience that the pupil-discovery method, although it may involve more time at the point of first teaching, works toward economy of time in the later stages of a course where more difficult relationships are being considered. With a stronger background of understandings discovered, the pupil finds himself in a better position to learn new material.

• WHAT IS GOING ON IN YOUR SCHOOL?

Edited by John A. Brown, University of Wisconsin, Madison, Wisconsin, and Houston T. Karnes, Louisiana State University, Baton Rouge, Louisiana

More on factoring the trinomial

by Sister Mary Dolorosa A., Cathedral High School, Portland, Maine

It was pleasing to read Mr. Gifford Welling's article on factoring the trinomial![1] I have used a similar teaching plan for this problem and offer the following explanation:

Example: Factor $3X^2-13X-10$

1. Multiply the first term by the last term.

$$(3X^2)(-10)=-30X^2$$

2. Find two factors of this product whose algebraic sum equals the middle term. Find the sign first. Since the sign of the product is minus, the sign of the terms will be unlike, and the sign of the greater will be the sign of the middle term.

$$(-15X)(2X)$$

3. Place these factors between the first and last terms.

$$(3X^2-5X)+(2X-10)$$

4. Group and factor by removing the monomial factor.

$$3X(X-5)+2(X-5)$$

5. Complete the factoring.

$$(3X+2)(X-5)$$

Note: If a quantity has no prime factor, one must be used.

Example: $X+5=1(X+5)$

Questions to ask pupils:

1. What is sign of product?
2. Then what will signs of factors be?
3. If unlike, what will sign of greater be?
4. If like, what will they be?
   Sign of middle term.
5. Why do we not bring down *middle term?*
   Pupils readily discern that middle term equals sum of factors.
   After Step 4 pupils see that terms inside ( ) must be same.

[1] "Factoring the Trinomial," The Mathematics Teacher, XLVIII (February 1955), p. 108.

It is toward the close of middle childhood that the power of abstract thinking begins to show itself. When a child can move from the concreteness of numbers in his arithmetic to the abstractness of allowing x to stand for any of a set of numbers, he not only is ready for algebra, he is ready for abstract thinking. When a child can assume something to be true for the sake of the argument when he knows it is probably not true, he is ready for abstract thinking. He is also ready for the type of proof in demonstrative geometry which starts by assuming something to be true that looks patently false (such as that the sum of the angles of a triangle is greater than two right angles), and then deduces consequences which are so absurd that the original assumption is disproved logically as well as empirically. The logical "therefore" enters into the child's idiom at this point.—*Taken from* Human Development and education *by Robert J. Havighurst (New York: Longmans, Green and Company, 1953), chap. 7, p. 90.*

• NOTES FROM THE WASHINGTON OFFICE

Edited by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D. C.

State representatives of The National Council of Teachers of Mathematics

During most of its existence the Council has depended upon a group of state representatives. These persons are something like dollar-a-year people, except that they do not even get the dollar. About all the recognition they get is to have their names printed once each year in THE MATHEMATICS TEACHER. The list of representatives and their addresses for the present school year is given below and on the following page.

The representatives are asked to serve the Council in four ways: (1) to sell memberships and subscriptions; (2) to publicize Council projects, activities, and publications; (3) to try to get professional articles and announcements printed in state journals and newspapers; and (4) to gather information when needed. The representative in your state is probably working harder for the Council than you realize. He deserves your help and support.

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"A minute I wrote (about 1917) for the Ballistic Office ended with the sentence 'Thus σ should be made as small as possible.' This did not appear in the printed minute. But P. J. Grigg said, 'What is that?' A speck in a blank space at the end proved to be the tiniest σ I have ever seen (the printers must have scoured London for it)."
—*Taken from J. E. Littlewood,* A Mathematician's Miscellany.

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The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D.C.

NCTM convention dates

ANNUAL MEETING
April 11–14, 1956
Schroeder Hotel, Milwaukee, Wisconsin
Margaret Joseph, 1504 N. Prospect Avenue, Milwaukee 2, Wisconsin

JOINT MEETING WITH NEA
July 2, 1956
Portland, Oregon
Lesta Hoel, Public Schools, Portland, Oregon

SUMMER MEETING
August 19–22, 1956
University of California, Los Angeles, California
Clifford Bell, University of California, Los Angeles 24, California

CHRISTMAS MEETING
December 27–29, 1956
Arkansas State College, Jonesboro, Arkansas
Lyle J. Dixon, Arkansas State College, State College, Arkansas

Other professional dates

Sixth Annual Conference for Teachers of Mathematics
April 20–21, 1956
Bowling Green University, Bowling Green, Ohio
Eugene Smith, University School, Columbus, Ohio

Women's Mathematics Club of Chicago and Vicinity
April 21, 1956
Mandel's Tea Room, State and Madison Streets, Chicago, Illinois
Jean Bryson, Waukegan Township High School, 425 Surf Street, Chicago 14, Illinois

Annual Spring Meeting of the Missouri Council of Teachers of Mathematics
April 21, 1956
Fontbonne College, St. Louis, Missouri
Margaret F. Willerding, Harris Teachers College, St. Louis, Missouri

Spring Conference of the Minnesota Council of Teachers of Mathematics
April 27–28, 1956
Coffman Memorial Union, University of Minnesota, Minneapolis, Minnesota
Frances S. Ek, 1134 Grand Avenue, St. Paul, Minnesota

Ninth Annual Conference on the Teaching of Elementary and Secondary Mathematics
April 28, 1956
Illinois State Normal University, Normal, Illinois
Clyde T. McCormick, Illinois State Normal University, Normal, Illinois

Spring Meeting of Mathematics Section of Maryland State Teachers Association
April 28, 1956
University of Maryland, College Park, Maryland
Helen Warren, Snow Hill, Maryland

Seventh Annual Conference of the Michigan Council of Teachers of Mathematics
May 4–6, 1956
St. Mary's Lake Camp, Battle Creek, Michigan
Miss Murel Kilpatrick, 114 N. Hamilton Street, Ypsilanti, Michigan

Fifth Annual Meeting of the Pennsylvania Council of Teachers of Mathematics

May 5, 1956
University of Pittsburgh, Pittsburgh, Pennsylvania
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Chicago Elementary Teachers' Mathematics Club

May 14, 1956
Toffenetti's Restaurant, 65 W. Monroe Street, Chicago, Illinois
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Mathematics Institute

June 4–15, 1956
University of Oklahoma, Norman, Oklahoma
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Mathematics Institute (Sponsored by Iowa State Teachers College and the National Science Foundation)

June 18–July 28, 1956
Iowa State Teachers College, Cedar Falls, Iowa
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June 20–July 3, 1956
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June 24–29, 1956
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Workshop in Visual Aids in Mathematics for Junior-Senior High School

June 25–July 13, 1956
State Teachers College, Millersville, Pennsylvania
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Institute on Secondary and Collegiate Mathematics

July 2–August 11, 1956
Williams College, Williamstown, Massachusetts
Donald E. Richmond, Department of Mathematics, Williams College, Williamstown, Massachusetts

Eighth Annual Mathematics Institute of the Association of Teachers of Mathematics in New England

August 16–23, 1956
Williams College, Williamstown, Massachusetts
Barbara B. Betts, 138 Norfolk Avenue, Swampscott, Massachusetts

Fourth Annual Mathematics Institute of the Florida Council of Teachers of Mathematics

August 27–28, 1956
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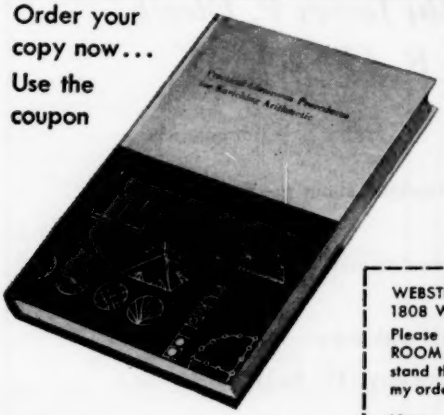

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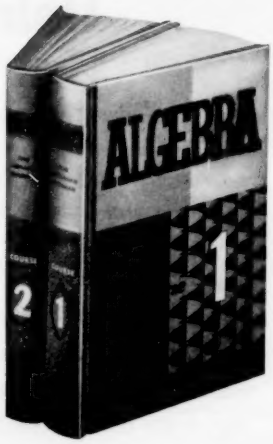

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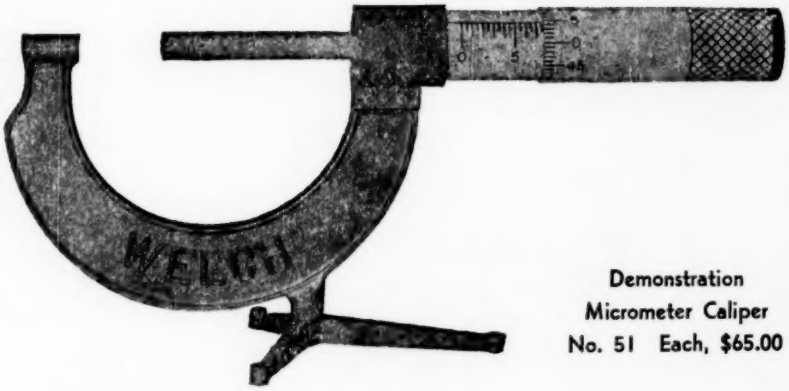

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